Invariant points, lines & planes (8 pages; 4/9/18)

(1) Overview

We will shortly see which of the following situations apply for the transformation $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$:

(a) Single invariant point

- this is the situation where none of the eigenvalues equals 1 (including cases where there are no eigenvalues)

- the Origin will be the only invariant point (see (8))

(b) Line of invariant points

- this situation corresponds to an eigenvalue of 1

- such lines will always pass through the Origin (see (9))

(c) Invariant line passing through the Origin

- points on one of these lines transform to other points (or the same point) on the line

- each such line corresponds to a particular eigenvalue

- a line of invariant points is a special case of an invariant line passing through the Origin, where the eigenvalue is 1

(d) Invariant lines NOT passing through the Origin

- these are not associated with eigenvectors (since the latter pass through the Origin)

(2) Suppose that $\begin{pmatrix} 2 & 4 \\ 3 & k \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$

Then $2p + 4q = p$

and $3p + kq = q$

so that $4q = -p$ & $(k-1)q = -3p$

Hence $\frac{q}{p} = -\frac{1}{4}$ & $\frac{q}{p} = -\frac{3}{k-1}$

so that $-\frac{1}{4} = -\frac{3}{k-1} \Rightarrow k - 1 = 12$ & hence $k = 13$

(3) Find the line of invariant points under the transformation given by the matrix $\begin{pmatrix} 2 & 4 \\ 3 & 13 \end{pmatrix}$

$\begin{pmatrix} 2 & 4 \\ 3 & 13 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$ (A)

$\Rightarrow 2p + 4q = p$ & $3p + 13q = q$

so that $4q = -p$ (or $12q = -3p$)

and hence $q = -\frac{p}{4}$

ie the invariant points lie on the line $y = -\frac{x}{4}$

Check: $\begin{pmatrix} 2 & 4 \\ 3 & 13 \end{pmatrix}\begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

(4) Alternative approach for (3) [this is the approach often used to find eigenvectors, once the eigenvalues have been established - in this case the eigenvalue is 1]

$\begin{pmatrix} 2 & 4 \\ 3 & 13 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 4 \\ 3 & 13 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}$

$\Rightarrow [\begin{pmatrix} 2 & 4 \\ 3 & 13 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}]\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 1 & 4 \\ 3 & 12 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow p + 4q = 0$

$\Rightarrow q = -\frac{p}{4}$

This is a line of invariant points through the Origin. It can be represented by the eigenvector $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$, corresponding to an eigenvalue of 1.

Every point on the line $y = -\frac{x}{4}$ is transformed to itself under the transformation $\begin{pmatrix} 2 & 4 \\ 3 & 13 \end{pmatrix}$.

Also, every point on this line is transformed to the point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ under the transformation $\begin{pmatrix} 1 & 4 \\ 3 & 12 \end{pmatrix}$ (which has a zero determinant).

(5) Find the line of invariant points under the transformation given by the matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$

$\Rightarrow -q = p$ (or $-p = q$)

So invariant points lie on the line $y = -x$

(as expected, as $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ represents a reflection in $y = -x$).

(6) Invariant lines passing through the Origin

For the transformation $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, we can apply the usual method of finding eigenvalues and then eigenvectors:

Let $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$

Then $\begin{pmatrix} 0-\lambda & -1 \\ -1 & 0-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (A)

and we require $\begin{vmatrix} 0-\lambda & -1 \\ -1 & 0-\lambda \end{vmatrix} = 0$, in order for (A) to have a solution in addition to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

So the characteristic equation is $(0-\lambda)(0-\lambda)-(-1)(-1)=0$,

giving $\lambda^2 = 1,$ and hence $\lambda = \pm 1$

$\lambda = 1 \Rightarrow \begin{pmatrix} 0-1 & -1 \\ -1 & 0-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

so that $y = -x$ (or eigenvector of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$)

whilst $\lambda = -1 \Rightarrow \begin{pmatrix} 0+1 & -1 \\ -1 & 0+1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

so that $y = x$ (or eigenvector of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$).

The line $y = -x$ was seen in (5) to be the line of invariant points.

(7) Invariant lines of a transformation (not necessarily passing through the Origin)

Consider lines $y = mx + c$ such that

$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} p \\ mp+c \end{pmatrix} = \begin{pmatrix} u \\ mu+c \end{pmatrix}$ **for all values of $\boldsymbol{p}$** (A)

Then $-(mp+c) = u$ & $-p = mu + c$

Eliminating u, $-p = m[-(mp+c)] + \mathrm{c}$

Hence $-m^2p - mc + c + p = 0$ (B)

We need to find values of m that satisfy this equation for all values of p.

Equating coefficients of powers of p:

$-m^2 + 1 = 0$ & $-mc + c = 0$;

ie $m = \pm 1$ and either $c = 0$ or $m = 1$

So the invariant lines are $y = x + c$ & $y = -x$

and these can be broken down into the categories referred to in (1):

(a) Single invariant point: n/a, as one of the eigenvalues is 1 (from (6))

(b) Line of invariant points: $y = -x$

(c) Invariant line passing through the Origin: $y = x$

(d) Invariant lines NOT passing through the Origin: $y = x + c$, where $c \neq 0$

(8) If none of the eigenvalues equal 1, then there is no solution to $M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$, apart from $x = y = 0$, so that the Origin is the only invariant point.

(9) Lines of invariant points will always pass through the Origin:

Such lines satisfy $M\binom{x}{y} = \binom{x}{y}$, and we know that, if such a line exists (when there is an eigenvalue of 1), then there will be a eigenvector representing that line. And since all eigenvectors correspond to lines through the Origin*, our line of invariant points will pass through the Origin.

[* Eigenvectors for 2×2 matrices, for example, are derived from an equation of the form $ax + by = 0$, for which the solution is of the form $y = mx$. For 3×3 matrices, the equations to be solved are of the form $ax + by + cz = 0$ and $dx + ey + fz = 0$, and the line of intersection of these planes passes through the Origin, since both planes contain the Origin.]

(10) Invariant planes for 3×3 transformations (and diagonalisability)

Having found the eigenvalues associated with a transformation, an invariant plane arises when the 3 simultaneous equations used to find the eigenvectors reduce to a single equation (typically in x, y & z); ie the equation of a plane. We can then choose any two non-parallel vectors in this plane as eigenvectors to cover the invariant plane.

In order for there to be an invariant plane, it can be shown that there must be repeated eigenvalues. But if there are repeated eigenvalues it doesn't follow that there will be an invariant plane (ie the repeated eigenvalue can just lead to an ordinary eigenvector - in other words, an invariant line).

The theory behind this is based on the following theorem: "The geometric multiplicity of an eigenvalue does not exceed its algebraic multiplicity." The algebraic multiplicity is the number of times that the eigenvalue appears as a root of the characteristic equation. The geometric multiplicity is the dimension of the line or plane relating to the eigenvalue: so an invariant line means a geometric multiplicity of 1, whilst an invariant plane means a geometric multiplicity of 2.

As an example of a situation where an eigenvalue is repeated but there is an invariant line, rather than an invariant plane, consider

$\begin{pmatrix} 3 & -1 & 1 \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{pmatrix}$, which has eigenvalues 2, 2 and -4.

The eigenvector associated with 2 turns out to be the line $y = x$ in the x-y plane (ie $z = 0$).

Note that, for this example, there are only 2 linearly independent eigenvectors, and so the matrix can't be diagonalised.

To reiterate: it isn't essential for the eigenvalues to be distinct, in order for the matrix to be diagonalisable. If two of the eigenvalues (for a 3×3 matrix) are the same, then the matrix will be diagonalisable if there is an invariant plane corresponding to the repeated eigenvalue. There will then be 2 eigenvectors covering the plane, and 3 (linearly independent) eigenvectors in total.

In general, A is not diagonalisable if, for some eigenvalue, the algebraic multiplicity (the number of equal eigenvalues) is greater than the geometric multiplicity (1 for an invariant line, 2 for an invariant plane etc).

Each eigenvalue has an 'eigenspace' associated with it; being the vector space covered by the eigenvectors associated with that eigenvalue. Thus, if a particular eigenvalue appears k times (ie the algebraic multiplicity is k), then the dimension of the eigenspace (which is the geometric multiplicity) will be $\leq k$. A matrix of order $n \times n$ is diagonalisable if and only if the sum of the dimensions of its eigenspaces equals n.