Further Integration Methods (STEP) (8 pages; 20/6/24)

See also: "Integration Methods".

Contents

(A) Substitutions

(1) The standard substitution method is to write an integral in the form $\int f(x)h\big(g(x)\big)\,dx$, where $\int f(x)dx = g(x)$, and then the substitution $u = g(x)$ will work, provided that $h(u)$ can be integrated.

In some cases it may be easier to spot a derivative, rather than an integral.

Consider $I = \int secx(secx + tanx)^n dx$ for example.

$\int secx\,dx = \ln\,|secx + tanx|$, which isn't of any use (ie the rest of the integrand $[(secx + tanx)^n]$ isn't a function of $\ln\,|secx + tanx|$ that can be integrated easily).

But if a substitution is to work, it will be $u = secx + tanx$,

and $\frac{d}{dx}(secx + tanx) = secxtanx + sec^2x$

Fortunately I can be rearranged to give

$\int(secx + tanx)^{n-1}(sec^2x + secxtanx)dx$,

and making the substitution $u = secx + tanx$ then gives

$I = \frac{1}{n}(secx + tanx)^n\ (+c)$

(2) $u = 1/x$ is a potentially useful substitution

Example 1: $I = \int \frac{1}{x\sqrt{1-x^2}} dx$

Let $u = 1/x$ so that $du = -1/x^2 dx$ and $dx = -x^2 du$,

so that $I = -\int \frac{ux^2}{\sqrt{1-\frac{1}{u^2}}}\ du = -\int \frac{u^2x^2}{\sqrt{u^2-1}}\ du$

$= -\int \frac{1}{\sqrt{u^2-1}}\ du = -arcoshu = -arcosh(\ 1/x)$

Example 2: Consider $I = \int \frac{1}{(a^2+x^2)^r} dx$

Let $t = \frac{a}{x}$, so that $dt = -\frac{a}{x^2} dx$

and $I = \int \frac{-\left(\frac{a}{t}\right)^2/a}{(a^2+(\frac{a}{t})^2)^r} dt = -a^{1-2r} \int \frac{t^{2r-2}}{(t^2+1)^r} dt$

If $r = \frac{3}{2}$, then $I = a^{-2} \int \frac{t}{(t^2+1)^{\frac{3}{2}}} dt$,

and we can then make the substitution $u = t^2$ (as $\int t\ dt = \frac{1}{2}t^2$,

and $\frac{1}{(t^2+1)^{\frac{3}{2}}}$ is an integrable function of t^2).

(3) Miscellaneous substitutions

(3.1) Example: $I = \int \frac{1}{x(a+bx^n)}\ dx$

Let $\frac{1}{z} = x^n$, so that $-\frac{1}{z^2}\ dz = nx^{n-1}dx$

Then $I = \int \frac{x^{n-1}}{x^n(a+bx^n)}\ dx = \int \frac{(-\frac{1}{nz^2})}{(\frac{1}{z})(a+\frac{b}{z})} dz$

$= -\frac{1}{n}\int \frac{1}{az+b}\ dz$

(3.2) Example: $I = \int \frac{1}{x\sqrt{a+bx^n}}\ dx$

Let $\frac{1}{z^2} = x^n$, so that $-\frac{2}{z^3}\ dz = nx^{n-1}dx$

Then $I = \int \frac{x^{n-1}}{x^n\sqrt{a+bx^n}}\ dx$

$= \int \frac{(-\frac{2}{nz^3})}{(\frac{1}{z^2})\sqrt{a+\frac{b}{z^2}}} dz$

$= -\frac{2}{n}\int \frac{1}{\sqrt{az^2+b}} dz$

(3.3) Example: $I = \int \frac{x^n}{\sqrt{a+bx}}\ dx$

Let $a + bx = z^2$, so that $b\ dx = 2z\ dz$

Then $I = \int \frac{(z^2-a)^n}{b^n z}\ .\frac{2z}{b}\ dz$

$= \frac{2}{b^{n+1}}\int (z^2 - a)^n dz$

(4) Pitfalls with substitutions

(4.1) $u = 1/x$ won't be valid for $x = 0$, but it can be applied in the case of $\int_0^2 \frac{1}{(1+x^2)^{\frac{3}{2}}}\ dx$, for example, by considering

$\lim_{c\to 0^+} \int_c^2 \frac{1}{(1+x^2)^{\frac{3}{2}}}\ dx$

(4.2) Consider $I = \int_{-2}^{-1} \frac{1}{(1+x^2)^{\frac{3}{2}}}\ dx$

With $x = \frac{1}{t}$, $dx = -\frac{1}{t^2} dt$,

$I = \int_{-2}^{-1} \frac{-\frac{1}{t^2}}{(1+\frac{1}{t^2})^{\frac{3}{2}}}\, dt$

But now note that, for the domain of this integral, $t < 0$,

so that we cannot rewrite $t^3(1 + \frac{1}{t^2})^{\frac{3}{2}}$ as $(t^2 + 1)^{\frac{3}{2}}$, because $(t^2 + 1)^{\frac{3}{2}} > 0$, whereas $t^3(1 + \frac{1}{t^2})^{\frac{3}{2}} < 0$ ($t^3 = t^{2\times\frac{3}{2}}$, but this doesn't equal $(t^2)^{\frac{3}{2}}$ if $t < 0$; in general, $t^{ab} = (t^a)^b$ is not valid for $t < 0$ unless both a & b are integers).

However, we can make the substitution $x = -\frac{1}{t}$ instead.

(4.3) Consider $I = \int_{-1}^{1} \frac{t}{(t^2+1)^{\frac{3}{2}}}\, dt$

The integrand is an odd function, and so I must equal zero. (See below: "Even and Odd functions".)

Writing $u = t^2, du = 2t\, dt$ gives $I = -\int_{\frac{1}{4}}^{\frac{1}{4}} \frac{\left(\frac{1}{2}\right)}{(u+1)^{\frac{3}{2}}}\, du = 0$, as expected; however, the substitution isn't valid for $t < 0$, as t^2 then doesn't increase as t increases.

(5) Substitutions in definite integrals

Look for a substitution that reverses the limits (and then take advantage of the fact that $\int_a^b f(x)dx = -\int_b^a f(x)dx$).

(i) $\int_0^\infty f(x)dx$: When $u = \frac{1}{x}$, $\int_0^\infty \;\to \int_\infty^0$ [though in practice we would need to consider $\lim_{c\to 0^+\ \&\ d\to\infty} \int_c^d f(x)\, dx$, as $u = \frac{1}{x}$ is undefined at $x = 0$ and the d is needed to cope with the improper integral.]

(ii) $\int_0^a f(x)dx$: When $u = a - x$, $\int_0^a \ \rightarrow \int_a^0$

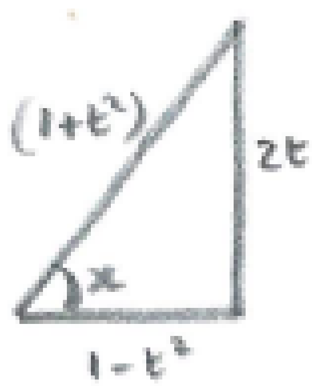
(6) Alternative substitutions

$sec\theta$ can often be used instead of $coshx$, and $tan\theta$ instead of $sinhx$.

(7) $t = \tan\left(\frac{x}{2}\right)$ substitution

The substitution $t = \tan\left(\frac{x}{2}\right)$ is usually a method of last resort: it can convert an integrand involving trig. functions to one involving polynomial expressions.

$t = \tan\left(\frac{x}{2}\right) \Rightarrow \ tanx = \frac{2t}{1-t^2}$



Referring to the right-angled triangle shown,

the hypotenuse = $\sqrt{(1-t^2)^2 + 4t^2}$

$= \sqrt{1 + 2t^2 + t^4} \ = 1+t^2$ (conveniently)

$\frac{dt}{dx} = sec^2\left(\frac{x}{2}\right).\frac{1}{2}$, so that $\frac{dx}{dt} = \frac{2}{sec^2\left(\frac{x}{2}\right)} = \frac{2}{1+t^2}$

Example: $\int secx \ dx = \int \frac{1+t^2}{1-t^2}.\frac{2}{1+t^2}$ dt $= 2\int \frac{1}{1-t^2} \ dt$

$= \int \frac{1}{1-t} + \frac{1}{1+t} \ dt$ = - ln|1-t| + ln|1+t| = ln$\left|\frac{1+t}{1-t}\right|$ = ln $\left|\frac{1+2t+t^2}{1-t^2}\right|$

$= \ln\left|\frac{1+t^2}{1-t^2} + \frac{2t}{1-t^2}\right| = \ln|secx + tanx|$

(B) Rearrangements

(1) It might be possible to rearrange an integrand into the form

$f(x)g'(x) + f'(x)g(x) + h(x)$, where $h(x)$ can be integrated easily, in which case $\int f(x)g'(x) + f'(x)g(x)\,dx = f(x)g(x)$

[from the product rule for differentiation, or integration by parts]

Example: $\int 2\sqrt{1+x^3} + \frac{3x^3}{\sqrt{1+x^3}}\,dx$

$\int 2\sqrt{1+x^3}\,dx = 2x\sqrt{1+x^3} - \int 2x.\frac{\frac{1}{2}(3x^2)}{\sqrt{1+x^3}}\,dx$ (by Parts),

so that $\int 2\sqrt{1+x^3} + \frac{3x^3}{\sqrt{1+x^3}}\,dx = 2x\sqrt{1+x^3} + c$

(2) Inequalities of the form $\int_a^\lambda f(x)dx > g(\lambda)$ can sometimes be proved by rewriting $g(\lambda)$ as $\int_a^\lambda h(x)dx$ (by differentiating $g(x)$ to obtain $h(x)$, if $g(a) = 0$) and then showing that

$\int_a^\lambda f(x) - h(x)\,dx > 0$, by rearranging $f(x) - h(x)$ into an expression that is positive for $a < x < \lambda$

(3) $\int \sin(mx)\cos(nx)\,dx = \frac{1}{2}\int \sin(m+n)\,x + \sin(m-n)\,x\,dx$

(C) Miscellaneous

(1) Even and odd functions

[An even function $f(x)$ is such that $f(-x) = f(x)$; an odd function is such that $f(-x) = -f(x)$.]

As $\frac{1}{(1+x^2)^{\frac{3}{2}}}$ is an even function, $\int_{-2}^{2}\frac{1}{(1+x^2)^{\frac{3}{2}}}\,dx = 2\int_0^2\frac{1}{(1+x^2)^{\frac{3}{2}}}\,dx$

As $\frac{x}{(1+x^2)^{\frac{3}{2}}}$ is an odd function, $\int_{-2}^{2}\frac{x}{(1+x^2)^{\frac{3}{2}}}\,dx = 0$ (if the area under the curve to the right of the y-axis is A, then the area to the left of the y-axis is $-A$).

(2) Questions that can be written in the form "Show that $\int_a^b f(x)dx = g(b) - c$" may be tackled by establishing that $\frac{d}{dx}g(x) = f(x)$ and that $g(a) = c$ (where typically a might equal 0).

(3) To find $\int f(x)dx = g(x)$, it might be the case that $g(x)$ appears in a previous part of a question. Differentiate $g(x)$ to see if this is the case. [See STEP 2016, P2, Q7(iv)]

(4) When manipulating an inequality involving an integral, it may be possible to simplify the integrand, as shown in the following example:

$\int_0^\lambda (secxcos\lambda + tanx)^n \, dx < \int_0^\lambda (secxcosx + tanx)^n \, dx$,

as $x < \lambda \Rightarrow cosx > cos\lambda$ (given that $0 < \lambda < \frac{\pi}{2}$),

$= \int_0^\lambda (1 + tanx)^n \, dx$

[See STEP 2021, P3, Q3]

(5) $\int_{-a}^{a} f(-x)\, dx = \int_{-a}^{a} f(x)\, dx$

Proof

Let $u = -x$, so that $du = -dx$, and

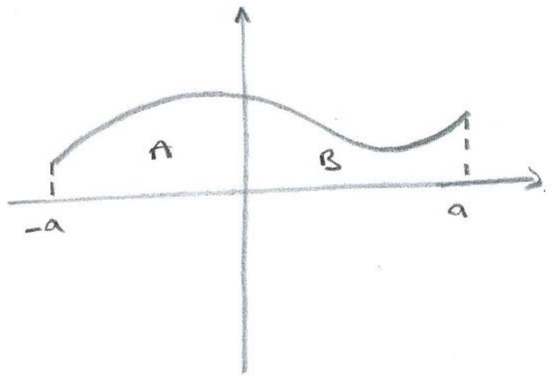
$\int_{-a}^{a} f(-x)\, dx = \int_{a}^{-a} f(u)(-du) = \int_{-a}^{a} f(u)\, du = \int_{-a}^{a} f(x)\, dx$

[Alternatively, considering the integral as an area under a curve, note that $f(-x)$ is the reflection of $f(x)$ about the y-axis, so that

$\int_{-a}^{0} f(-x)\, dx = B = \int_{0}^{a} f(x)\, dx$ (referring to the diagram below)

and $\int_{0}^{a} f(-x)\, dx = A = \int_{-a}^{0} f(x)\, dx$,

so that $\int_{-a}^{a} f(-x)\, dx = B + A = A + B = \int_{-a}^{a} f(x)\, dx$



(6) $\int_{0}^{a} f(\mathrm{a} - x)\, dx = \int_{0}^{a} f(x)\, dx$

Proof

Let $u = a - x$, so that $du = -dx$ and

$\int_{0}^{a} f(\mathrm{a} - x)\, dx = \int_{\mathrm{a}}^{0} f(\mathrm{u})\,(-\mathrm{du}) = \int_{0}^{a} f(u)\, du = \int_{0}^{a} f(x)\, dx$

[Note that $f(\mathrm{a} - x)$ is the reflection of $f(x)$ about $x = \frac{a}{2}$.]

(7) To find $\int cosech^2 x\, dx$, note that $\frac{d}{dx}(tanhx) = sech^2 x$ and establish that $\frac{d}{dx}(cothx) = -cosech^2 x$