Integration Methods (11 pages; 15/6/24)

[The constant of integration is generally omitted in these notes.]

See also Pure: "Integration Exercises" (Parts 1-4) and STEP: "Further Integration Methods".

Contents

(A) Substitutions

(1) Substitutions may be considered to be of the following types (with possible overlap):

(a) A linear substitution (ie of the form $u = ax + b$) which will simplify the integrand (the expression being integrated), without introducing any unwanted complications from the change of variable (as $du = adx$).

(b) A 'speculative' substitution, which has a good chance of simplifying the integrand, even after the complications from the change of variable.

(c) Integrals of the form $\int f(x)h\big(g(x)\big)\,dx$, where

$\int f(x)dx = g(x)$, where it will be seen that the substitution

$u = g(x)$ will work, provided that $h(u)$ can be integrated.

(2) Linear substitutions

Example: $I = \int x(1+x)^{\frac{1}{2}}\,dx$

Note that $(1+x)^{\frac{1}{2}}$ can only be expanded as an infinite Binomial series. However, let $u = 1 + x$, giving $I = \int (u-1)u^{\frac{1}{2}}\, du$, and the integrand can now be expanded.

(3) 'Speculative' substitutions

Example: $I = \int \sqrt{1-x^2}dx$

Let $x = sin\theta,$ so that $dx = cos\theta\ d\theta$

Then $I = \int cos\theta cos\theta d\theta = \frac{1}{2}\int 1 + cos2\theta\ d\theta$

$= \frac{1}{2}\theta + \frac{1}{4}sin2\theta = \frac{1}{2}\theta + \frac{1}{2}sin\theta cos\theta$

$= \frac{1}{2}arcsinx + \frac{1}{2}x\sqrt{1-x^2}$

(as can be confirmed by differentiation)

Example: $I = \int \sqrt{x^2+2x+5}\, dx = \int \sqrt{(x+1)^2+2^2}\, dx$

Let $x + 1 = 2sinhy$, so that $dx = 2coshy\ dy$

and $I = 2\int \sqrt{sinh^2y+1}\,(2coshy)\, dy$

$= 4\int cosh^2y\ dy = 2\int 1 + \cosh(2y)\, dy$

$= 2y + \sinh\,(2y)$

$= 2arsinh\left(\frac{x+1}{2}\right) + 2sinhycoshy$

$= 2arsinh\left(\frac{x+1}{2}\right) + (x+1)\sqrt{sinh^2y+1}$

$= 2arsinh\left(\frac{x+1}{2}\right) + (x+1)\sqrt{\left(\frac{x+1}{2}\right)^2+1}$

Example: $I = \int \sqrt{x^2+8x+7}\, dx = \int \sqrt{(x+4)^2-3^2}\, dx$

Let $x + 4 = 3coshy$, so that $dx = 3sinhy\ dy$

and $I = 3\int\sqrt{cosh^2y-1}\,(3sinhy)dy$

$= 9\int sinh^2y\,dy$

$= \frac{9}{2}\int \cosh(2y) - 1)\,dy$

(and then similarly to the previous example)

(4) For integrals of the form $I = \int f(x)h\big(g(x)\big)\,dx$, where $\int f(x)dx = ag(x)$, make the substitution $u = g(x)$ if $\int h(u)du$ can be determined.

Then $du = g'(x)\,dx = \frac{1}{a}f(x)\,dx$, so that $I = a\int h(u)\,du$

Note: $f(x)$ has to be in the numerator of the integrand

Example: $I = \int \tan x\;\;dx\;\;=\;\;\int\frac{\sin x}{\cos x}\;\;dx$

Integrating $sinx$ to give $-cosx$ reveals that the substitution $u = cosx$ will work:

$du = -sinx\,dx$, so that

$I = -\int\frac{1}{u}du = -lnu = -\ln(cosx) = \ln\,(secx)$

Example: $I = \int sinxcos^2x\,dx$

Noting that $\int sinx\,dx = -cosx$, let $u = cosx$,

so that $du = -sinx\,dx$, and $I = -\int u^2\,du$

Example: $I = \int\frac{sinx}{cos^2x}dx$

The substitution $u = cosx$ works here as well,

giving $I = -\int u^{-2}\,du$

Example: $\int \frac{e^x}{e^{2x}+1} dx$

Here $\int e^x dx = e^x$, and we can integrate $\int \frac{1}{u^2+1}\ du$, so let $u = e^x$.

Example: $I = \int \frac{1}{xlnx}\ dx$

Noting that $\int \frac{1}{x} dx = lnx$, let $u = lnx$, so that $du = \frac{1}{x} dx$,

and $I = \int \frac{1}{u}\ du = ln\ (lnu)$

(B) Rearrangements

Example: $\int \frac{1+x}{x-1}\ dx\ = \int \frac{x-1}{x-1}\ dx\ + \int \frac{2}{x-1}\ \ dx$ etc

Example: $\int \frac{1}{1+cosx}\ dx = \int \frac{1-cosx}{1-cos^2x}\ dx = \int \frac{1-cosx}{sin^2x}\ dx$

$= \int cosec^2x\, dx - \int \frac{cosx}{sin^2x}\ dx$ (and these can both be determined)

Example: $\int \sqrt{1 + sin2x}\ dx$

$1 + sin2x = 1 + cos\left(\frac{\pi}{2} - 2x\right) = 1 + cos2\left(\frac{\pi}{4} - x\right)$

$= 2cos^2(\frac{\pi}{4} - x)$

Example: $\int tan^2\theta\ d\theta = \int sec^2\theta - 1\ d\theta = tan\theta - \theta$

Example: $I = \int sec^4\theta\ d\theta = \int sec^2\theta(1 + tan^2\theta)d\theta$

[Spotting that $\int sec^2\theta\ d\theta = tan\theta$] Let $u = tan\theta$,

so that $du = sec^2\theta\ d\theta$, and $I = \int 1 + u^2\ du$

Example $\int sechxtanhx\,dx = \int \frac{sinhx}{cosh^2x}dx$

and then let $u = coshx$

In the case of hyperbolic functions, there is always the option of invoking the definition in terms of the exponential function.

Example: $\int sechxdx = \int \frac{2}{e^x+e^{-x}}dx = 2\int \frac{e^x}{e^{2x}+1}dx$

Then let $u = e^x$, to give $2\int \frac{1}{u^2+1}du = 2arctan(e^x)$

Example: $I = \int \frac{1}{x^4+1}dx = \int \frac{x^{-2}}{x^2+x^{-2}}\ dx$

[As a speculative substitution] Let $x = e^y$, so that $dx = e^y dy$ and $x^{-1}dx = dy$

Then $I = \int \frac{e^{-y}}{e^{2y}+e^{-2y}}dy = \frac{1}{2}\int \frac{e^{-y}}{\frac{1}{2}(e^{2y}+e^{-2y})}dy = \frac{1}{2}\int \frac{coshy-sinhy}{\cosh(2y)}dy$

$= \frac{1}{2}\int \frac{coshy}{2sinh^2y+1}dy - \frac{1}{2}\int \frac{sinhy}{2cosh^2y-1}dy$

(then let $u = sinhy$ & $u = coshy$, respectively).

(C) Integration by Parts

(1) This is an obvious option whenever the integrand is a product of two expressions (or can be rearranged into this form). However, substitution will also a possibility, and may well be preferable - especially as there are potential drawbacks with Parts:

(i) It may not be immediately clear which of the two expressions in the product needs to be integrated.

(ii) Applying Parts may not result in an improvement.

(iii) Parts may not work, or we may end up where we started (though there are steps that can be taken to avoid this - see below).

(iv) The process may be time-consuming if Parts has to be applied twice.

(2) The Parts formulae is derived from the product rule for differentiation:

$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$, where u & v are functions of x

Integrating both sides then gives

$uv = \int \frac{du}{dx}v\, dx + \int u\frac{dv}{dx}\, dx$

Then $\int \frac{du}{dx}v\, dx = uv - \int u\frac{dv}{dx}\, dx$

However, there is no need to write out all these symbols: note that one of the expressions in the product ($\frac{du}{dx}$) is being integrated (to u) and that the integrated expression appears in both the terms on the RHS. The expression that is to be differentiated (v) stays the same for the 1st term on the RHS, and is differentiated (to $\frac{dv}{dx}$) only in the 2nd term. In order not to get confused, the two expressions in the original integral could be labelled with an I and a D (indicating that they are to be integrated and differentiated, respectively).

(3) Standard situation

Example $I = \int x\, sinx\, dx$

In order to obtain a simpler integral on the RHS, we generally want to reduce the power of a term such as x^n (but see below for

an exception). So x^n should generally be differentiated.
$sinkx, coskx$ & e^{kx} can be integrated or differentiated.

Here $I = x(-cosx) - \int (1)(-cosx)dx$ etc

Example $I = \int x(3x+1)^3 dx$

Integrating $(3x+1)^3$ and differentiating x,

$I = x\left(\frac{1}{4}\right)(3x+1)^4\left(\frac{1}{3}\right) - \int (1)\left(\frac{1}{4}\right)(3x+1)^4\left(\frac{1}{3}\right)dx$

$= \frac{1}{12}x(3x+1)^4 - \frac{1}{12}\int (3x+1)^4 dx$

$= \frac{1}{12}x(3x+1)^4 - \frac{1}{12}\left(\frac{1}{5}\right)(3x+1)^5\left(\frac{1}{3}\right)$

$= \frac{(3x+1)^4}{180}\{15x - (3x+1)\} = \frac{(3x+1)^4(12x-1)}{180}$

Example: $I = \int sinxcosx\ dx$

Integrating $cosx$ and differentiating $sinx$ (to avoid unnecessary minus signs):

$I = \int sinx\ cosx\ dx = sinx.sinx - \int cosx.sinx\ dx = sin^2x - I$

Hence $I = \frac{1}{2}sin^2x$

Example: $\int \frac{lnx}{x}\ dx$

Integrating $\frac{1}{x}$:

$I = \int \frac{lnx}{x}\ dx = lnx.lnx - \int lnx.\frac{1}{x}\ dx = (lnx)^2 - I$

Hence $I = \frac{1}{2}(lnx)^2$

Example: $\int \ln x \, dx$

write as $\int 1 \, . \ln x \, dx$

Integrating 1,

$I = xlnx - \int x(1/x)dx = xlnx - x$

(4) When applying Parts twice, the function resulting from integrating one of the components always has to be integrated again (to avoid going round in circles).

Example: $I = \int sinx \, . e^x \, dx$

Integrating e^x, for example,

$I = sinxe^x - \int cosxe^x dx$

Then integrating e^x again, to apply Parts for a 2nd time:

$I = sinxe^x - \{cosxe^x - \int (-sinx)e^x \, dx\}$

$= (sinx - cosx)e^x - I$

so that $I = \frac{1}{2}(sinx - cosx)e^x$

[If we had chosen to differentiate e^x, this would have given

$I = sinxe^x - \{sinxe^x - \int sinxe^x \, dx\} = I$]

(5) Definite integral

Example: $I = \int_1^e \left(\frac{lnx}{x}\right)^2 dx$

Writing the integrand as $\frac{1}{x^2}(lnx)^2$ and differentiating $(lnx)^2$:

$I = \left[-\frac{1}{x}\,(lnx)^2\right]_1^e - \int_1^e \left(-\frac{1}{x}\right) 2lnx \left(\frac{1}{x}\right) dx$

$= -\left(\frac{1}{e} - 0\right) + 2\int_1^e \frac{lnx}{x^2} dx$

Then differentiating lnx, to apply Parts again:

$I = -\frac{1}{e} + 2\left[-\frac{1}{x}lnx\right]_1^e - 2\int_1^e \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right) dx$

$= -\frac{1}{e} - 2\left(\frac{1}{e} - 0\right) + 2\int_1^e \frac{1}{x^2} dx$

$= -\frac{3}{e} + 2\left[-\frac{1}{x}\right]_1^e = -\frac{3}{e} - 2\left(\frac{1}{e} - 1\right) = 2 - \frac{5}{e}$

(6) Parts doesn't always work:

Example: $\int \frac{1}{xlnx}\ dx$

Differentiating $\frac{1}{lnx}$:

$I = lnx\,.\frac{1}{lnx} - \int lnx\ .\ (-1)(lnx)^{-2}\left(\frac{1}{x}\right) dx$

$= 1 + I$?!

The apparent contradiction is due to the constant of integration.

Thus Parts can't be used to find this integral, but the substitution $u = lnx$ works , as already seen.

(D) Reduction formulae

Integration by Parts can sometimes enable a recurrence relation to be set up.

Example: $I_n = \int_0^1 x^n e^{-x}\, dx$

Integrating e^{-x} and differentiating x^n gives:

$I_n = [\ -e^{-x}\, x^n\ ]_0^1\ - \int_0^1 -nx^{n-1}e^{-x}\, dx$

$= \ -e^{-1} + 0\ + nI_{n-1}$

Thus $\ I_n = \ nI_{n-1} - e^{-1}$

Then, since $I_0 = \ \int_0^1 e^{-x}\, dx = [\,-e^{-x}\,]_0^1 \ = \ -e^{-1} + 1 = \ 1 - e^{-1}$,

$I_1 = \ (1 - e^{-1}) \ -e^{-1} \ = \ 1 - 2e^{-1},$

$I_2 = \ 2(1 - 2e^{-1}) - e^{-1} \ = 2 - 5e^{-1}$ etc

Example: $I_n \ = \int_0^{\pi} cos^n x \ \ dx$

Integrating by Parts (writing as $cosx \ . \ cos^{n-1}x$ and differentiating $cos^{n-1}x$)

leads to $I_n = \ \frac{n-1}{n} I_{n-2}$

Hence $\int_0^{\pi} cos^4 x \ = \ \frac{3}{4} \ I_2 \ = \ \frac{3}{4} \ . \ \frac{1}{2} \ \int_0^{\pi} 1 \ dx = \frac{3\pi}{8}$

Example: $I_n = \int_0^1 \frac{x^n}{\sqrt{1-x}} \ dx$

$I_n = \ [x^n \ . \ \frac{(1-x)^{1/2}}{-1/2} \ ]\begin{matrix} 1 \\ 0 \end{matrix} \ - \int_0^1 n x^{n-1} \frac{(1-x)^{1/2}}{-1/2} dx$

$= 0 + 2n \int_0^1 x^{n-1} . \ \frac{(1-x)}{(1-x)^{1/2}} \ \ dx$

(forcing the integrand into the form of I_n)

$= 2n(I_{n-1} - \ I_n)$

$\Rightarrow \ (1 + 2n) I_n = 2n I_{n-1}$

$\Rightarrow \ I_n = \ \frac{2n I_{n-1}}{2n+1}$

Notes

(i) Reduction formulae can also be derived for indefinite integrals.

(ii) Sometimes the integrand can be rearranged to give the reduction formula, without performing integration by Parts

eg $\int tan^n x dx = \int tan^{n-2} x (sec^2 x - 1) dx$

$= \frac{1}{n-1} tan^{n-1} x - \int tan^{n-2} x \ dx$ (as $sec^2 x$ is the derivative of $tanx$)

(iii) Reduction formulae are often associated with a proof by induction.

(iv) If $I_n = \cdots I_{n-2}$, the result will depend on whether n is odd or even.