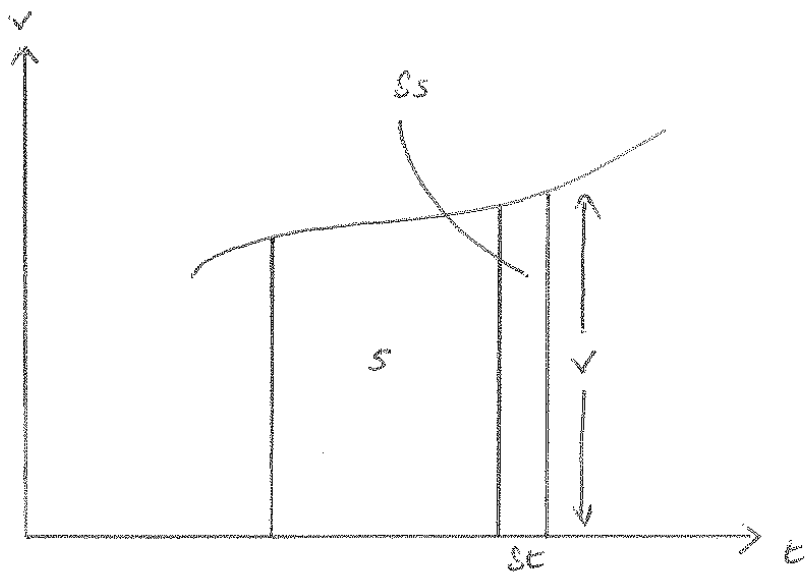
Integration Theory (7 pages; 7/11/24)

Contents

(A) The two interpretations of integration

Integration can be interpreted as either the area under a curve, or as the opposite of differentiation. To show how these two interpretations can be reconciled, refer to the diagram below.



v & s can be interpreted as speed and displacement, but the argument holds for other situations. s is defined to be the area under the curve of v, and, by the first definition of integration,

$s = \int v\,dt$ (A)

(to work out a specific area, limits would obviously be needed).

We want to show that integration is also the opposite of differentiation. This will be the case if $\frac{ds}{dt} = v$

From the diagram, $\frac{ds}{dt}$ is the rate at which the area increases, and is the limit as $\delta t \to 0$ of $\frac{\delta s}{\delta t}$, which equals v, since $\delta s \to v\delta t$ as $\delta t \to 0$. Thus we have shown that $\frac{ds}{dt} = v$.

In the case where v & s are speed and displacement, this works because speed is the rate of change of displacement, and displacement = speed × time if the speed is constant (so that the displacement is the area under a horizontal line), and the natural extension of this is for the displacement to be the area under the speed-time graph in the case of a varying speed.

(B) Indefinite integration

In the definite integral $\int_{t_1}^{t_2} v(t)dt$, t is appearing as a parameter (which ranges from t_1 to t_2). It can just as easily be written as

$\int_{t_1}^{t_2} v(x)dx$

If t_2 is now considered to be a variable value of t, so that the definite integral represents the area under the curve as a function of t_2, then, writing t instead of t_2: $\int_{t_1}^{t} v(x)dx = s(t) - s(t_1)$

(where $v(x)$ is the derivative of $s(x)$; eg speed and displacement, respectively).

The integral is now a function of t (whereas the definite integral $\int_{t_1}^{t_2} v(t)dt$ was a fixed value).

It is termed an 'indefinite' integral and, by convention, the following notation is adopted: $\int v(t)dt = s(t) + C$

C in effect equals $-s(t_1)$ and is a constant ; ie not changing with t (C is the 'constant of integration'). It can take any value (including positive values, since $s(t_1)$ can generally be made to be negative).

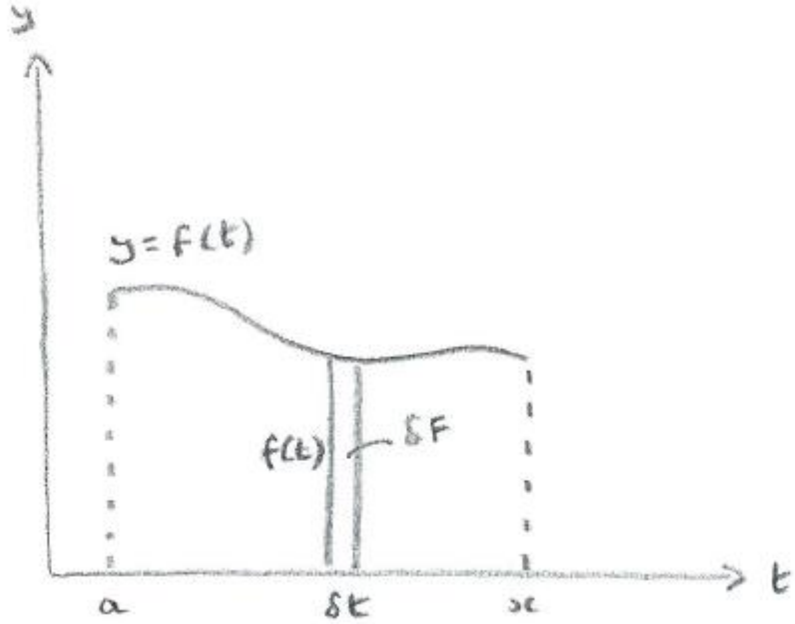
Note that t has been reintroduced on the left hand side, as it can no longer be confused with the upper limit of integration. This notation is slightly unsatisfactory, since the t on the left hand side is a parameter over which the integration is being carried out, whereas the t on the right hand side is the upper limit of the integration. However, the t on the left hand side does serve to indicate that the integral is to be a function of t.

(C) Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus states that

if $F(x) = \int_a^x f(t)\, dt$, then $F'(x) = f(x)$

Proof



$\delta F \approx f(t)\delta t \Rightarrow \frac{\delta F}{\delta t} \approx f(t)$

$F'(t)\ or\ \frac{dF}{dt} = \lim_{\delta t \to 0} \frac{\delta F}{\delta t} = f(t)$

and at $t = x,\ F'(x) = f(x)$

(D) $\int \frac{1}{x} dx = \ln|x|$

Given that $\int \frac{1}{x} dx = \ln x$ for $x > 0$, it can be shown that $\int \frac{1}{x} dx = \ln|x|$ for all $x \neq 0$

Method 1

If $\int \frac{1}{x} dx = \ln x$ for $x > 0$, then $\frac{d}{dx}(lnx) = \frac{1}{x}$ for $x > 0$

For the case where $x < 0$:

Let $y = -x$, so that $\frac{d}{dy}(lny) = \frac{1}{y}$, as $y > 0$

[To convert back to xs:]

Then, as $\frac{d}{dy}(lny) = \frac{d}{dx}(lny).\frac{dx}{dy}$,

it follows that $\frac{d}{dx}(lny).\frac{dx}{dy} = \frac{1}{(-x)}$

giving $\frac{d}{dx}(ln[-x])(-1) = \frac{1}{(-x)}$

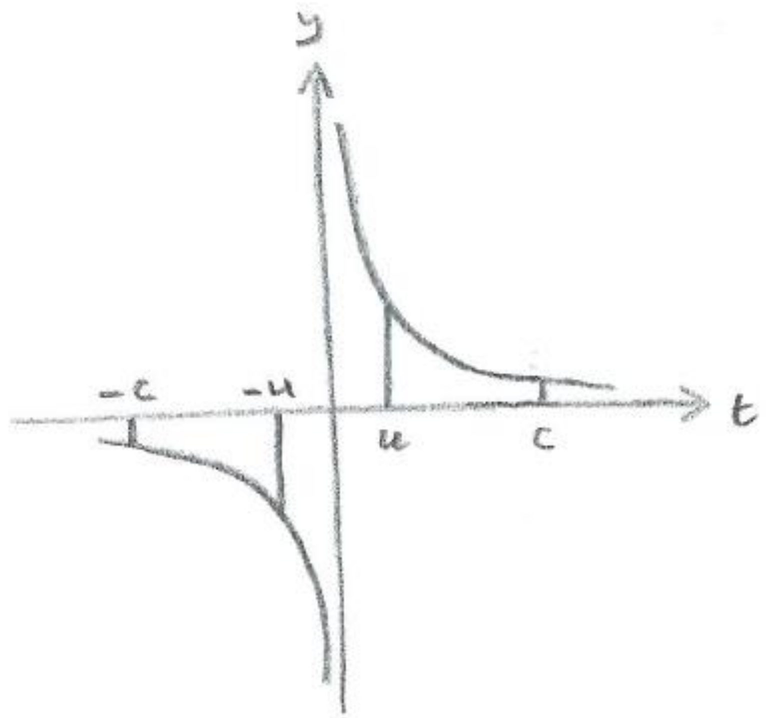
and so $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$ for $x < 0$ (*)

and therefore $\int \frac{1}{x}dx = \ln|x|$ for $x < 0$, as well as $x > 0$

[Note that the function $y = \ln\ |x|$ for $x < 0$ is the reflection in the y-axis of $y = \ln x$ $(for\ x > 0)$, and therefore has a negative gradient, which agrees with (*).]

Method 2

Referring to the diagram below, where $u = -x > 0$ & $c > 0$,



$\int_{-c}^{x} \frac{1}{t} dt = \int_{-c}^{-u} \frac{1}{t} dt$

$= -$ (positive) area between graph and t-axis on LHS

$= -$ (positive) area between graph and t-axis on RHS

$= -\int_{u}^{c} \frac{1}{t} dt = \int_{c}^{u} \frac{1}{t} dt = lnu - lnc$

As $\int \frac{1}{x} dx$ only differs from $\int_{-c}^{x} \frac{1}{t} dt$ by an arbitrary constant, it follows that, when $x < 0$, $\int \frac{1}{x} dx = ln\, u + C = \ln|-x| + C$, as required.

(E) Substitutions

Example: $I = \int \frac{sinx}{cosx}\ dx$

Informal approach (but acceptable for exam purposes)

Let $u = cosx$, so that $du = -sinx\ dx$,

and then $I = -\int \frac{1}{u}\ du = -\ln u + C = -\ln(cosx) + C$

$= \ln(secx) + C$

More rigorous approach

Let $u = cosx$

[Result to prove: $I = \int \frac{sinx}{cosx} \frac{dx}{du}\ du$;

then, as $\frac{dx}{du} = \frac{1}{(\frac{du}{dx})} = \frac{1}{-sinx}$, $I = -\int \frac{1}{u}\ du$]

Now, $I = \int \frac{sinx}{cosx}\ dx \Rightarrow \frac{dI}{dx} = \frac{sinx}{cosx}$

and $\frac{dI}{du} = \frac{dI}{dx} . \frac{dx}{du} = \frac{sinx}{cosx} . \frac{dx}{du}$, so that $I = \int \frac{sinx}{cosx} \frac{dx}{du}\ du$

Then $\frac{dx}{du} = \frac{1}{(\frac{du}{dx})} = \frac{1}{-sinx}$,

so that $I = -\int \frac{1}{u}\ du$ etc