2011 MAT Paper - Multiple Choice (7 pages; 6/10/24)

Q1/A

Solution

(a) starts in the wrong quadrant, and so can be eliminated.

If $f(x) = x^3 - x^2 - x + 1$,

$f'(x) = 3x^2 - 2x - 1 = (x - 1)(3x + 1)$

[Had $(x - 1)$ not been a factor, (c) could have been eliminated.]

Thus there is a stationary point at $x = 1$,

and so **the answer must be (c)**, by elimination.

[Alternative approaches:

(i) Consider x-coordinate of point of inflexion $(-\frac{b}{3a} = \frac{1}{3})$

(ii) Sum of roots is expected to be $-\frac{b}{a} = 1$, which rules out (a) & (b), and is consistent with (c). However, there are two (as yet unknown) complex roots for (d).]

Q1/B

Solution

Let the sides of the rectangle be x & y. Then $P = 2(x + y)$ and $A = xy$.

Of the 4 possibilities, (c) is the first one that looks feasible.

Then $P^2 - 16A = 4(x^2 + y^2 + 2xy) - 4(4xy) = 4(x - y)^2 \geq 0$

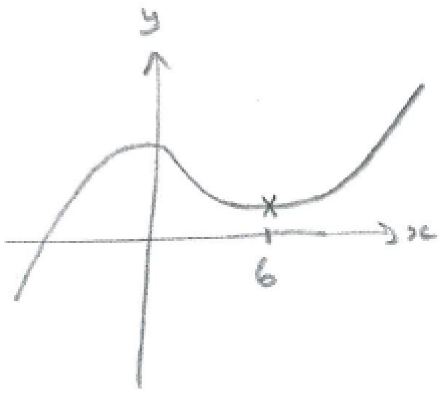
So the answer is (c).

Q1/C

Solution

Let $y = x^3 - 9x^2 + 631$.

Then $\frac{dy}{dx} = 0 \Rightarrow 3x^2 - 18x = 0 \Rightarrow x = 0 \; or \; 3x - 18 = 0; \; ie \; x = 6$



Thus, $x = 5$ is the largest integer, for which $y(x) > y(x+1)$.

So the answer is (a).

Notes

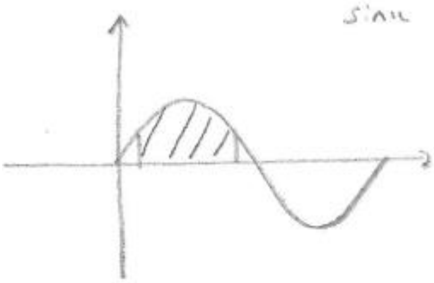
The official sol'ns are missing a " $>$ " sign; ie $-3n^2 + 15n + 8 > 0$

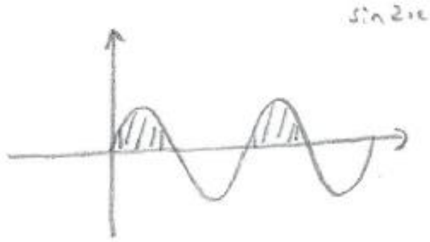
Using the 'official' method, to be absolutely sure that $n = 5$ is the largest value, we can set $\frac{15+\sqrt{A}}{6} = 5$; which gives $A = 225$, so that

$\frac{15+\sqrt{321}}{6} > 5$ (and already shown to be < 5.5), whilst $\frac{15-\sqrt{321}}{6} < 5$

Q1/D

Solution





One or both of the inequalities is true for the following regions:

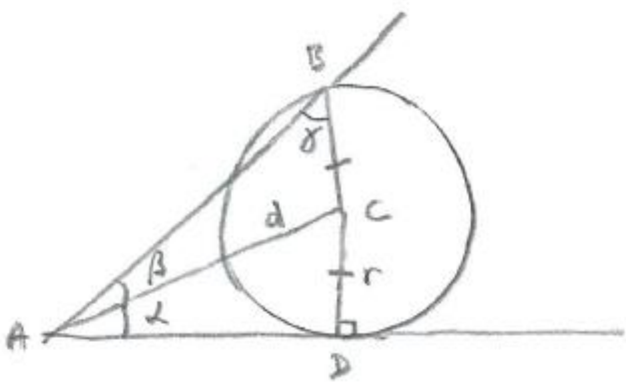
$\left[\frac{\pi}{12}, \frac{5\pi}{6}\right]$ & $[\pi + \frac{\pi}{12}, \pi + \frac{5\pi}{12}]$

So the required proportion is $\frac{\left(\frac{(9+4)\pi}{12}\right)}{2\pi} = \frac{13}{24}$

So the answer is (b).

Q1/E

Solution



Drawing in the radius CD, as in the diagram,

$sin\alpha = \frac{r}{d}$ & $\frac{sin\beta}{r} = \frac{sin\gamma}{d}$

so that $sin\alpha = \frac{sin\beta}{sin\gamma}$ or $sin\beta = sin\alpha sin\gamma$

So the answer is (b).

Q1/F

Solution

$x^2 + y^2 + 4xcos\theta + 8ysin\theta + 10 = 0$

$\Rightarrow (x + 2cos\theta)^2 - 4cos^2\theta + (y + 4sin\theta)^2 - 16sin^2\theta + 10 = 0$

Then, for a circle of radius r, $r^2 = 4cos^2\theta + 16sin^2\theta - 10$

$= 4 + 12sin^2\theta - 10$

$= 12sin^2\theta - 6$

And $12sin^2\theta - 6 > 0 \Rightarrow sin^2\theta > \frac{1}{2}$,

so that, for $0 \leq \theta < \pi, sin\theta > \frac{1}{\sqrt{2}}$

$\Rightarrow \frac{\pi}{4} < \theta < \frac{3\pi}{4}$

So the answer is (b).

Q1/G

Solution

Note that, for $-1 \leq x \leq 1, 0 \leq x^2 \leq 1$

so that $-1 \leq x^2 - 1 \leq 0$

So $f(x^2 - 1) = (x^2 - 1) + 1$ (since the equation of the left-hand sloping part of the graph is $y = x + 1$)

and hence $\int_{-1}^{1} f(x^2 - 1)dx = \int_{-1}^{1} x^2\, dx = \left[\frac{1}{3}x^3\right]_{-1}^{1} = \frac{2}{3}$

So the answer is (d).

Q1/H

Solution

$x = 8^{log_2 x} - 9^{log_3 x} - 4^{log_2 x} + log_{0.5}0.25$

$= 2^{3log_2 x} - 3^{2log_3 x} - 2^{2log_2 x} + 2$

$= (2^{log_2 x})^3 - (3^{log_3 x})^2 - (2^{log_2 x})^2 + 2$

$\Rightarrow x = x^3 - x^2 - x^2 + 2$

$\Rightarrow x^3 - 2x^2 - x + 2 = 0$

[Unsurprisingly,] $x = 1$ is a root,

giving $(x - 1)(x^2 - x - 2) = 0$,

so that $(x - 1)(x + 1)(x - 2) = 0$

and there are two positive values of x.

So the answer is (c).

Q1/I

Solution

Let $y = sin^2 x$, so that $y^4 + (1 - y)^3 = 1$

$\Rightarrow y^4 - y^3 + 3y^2 - 3y = 0$

$\Rightarrow y(y^3 - y^2 + 3y - 3) = 0$

$\Rightarrow y(y - 1)(y^2 + 3) = 0$

$\Rightarrow sin^2x = 0 \; or \; 1$

$\Rightarrow x = 0, \pi, \frac{\pi}{2} \; or \; \frac{3\pi}{2}$

So the answer is (b).

Q1/J

Solution

$f(1) = 1$

$f(2) = f(1) = 1$

$f(3) = [f(1)]^2 - 2 = -1$

$f(4) = f(2) = 1$

$f(5) = [f(2)]^2 - 2 = -1$

$f(6) = f(3) = -1$

$f(7) = [f(3)]^2 - 2 = -1$

$f(8) = f(4) = 1$

Thus $f(n) = -1$ when n is odd (except for $n = 1$).

For even n, $f(n) = 1$ if n is a power of 2,

and $f(n) = -1$ if n is of the form $(2^k)(2p + 1)$ (ie all other even numbers)

Of the 100 values being added,

$n = 1 \Rightarrow 1$ [1 value]

other odd $\Rightarrow -1$ [49 values]

powers of 2 $\Rightarrow 1$ [6 values]

other even numbers $\Rightarrow -1$ [44 values]

So $f(1) + f(2) + f(3) + \cdots + f(100)$

$= (1 + 6) - (49 + 44) = -86$

So the answer is (a).