2023 MAT - Multiple Choice (8 pages; 19/7/24)

Q1/A

(a) $2\beta = log(9)$ (all logs are to base 10)

(b) $5\alpha + \beta = log(32 \times 3) = log(96)$

(c) $\alpha + 2\gamma = log(2 \times 49) = log(98)$

(d) $2\alpha + 5\beta = log(4 \times 243) = log(972)$

(e) $2\alpha + \beta + \gamma = log(4 \times 3 \times 7) = log(84)$

[Referring to the hint:]

$log(972) = log(9.7 \times 100) = log(9.7) + 2$

Of the 5 options, $log(98) = log(9.8) + 1$ is closest to $log(10)$ when the integer part is deducted

[Strictly speaking, we should establish that $log(8.4)$ is further from $log(1)$ than $log(9.8)$ is from $log(10)$;

ie that $log(8.4) - log(1) > log(10) - log(9.8)$;

or $log(8.4) + log(9.8) > log(10)$;

or $log(8.4 \times 9.8) > log(10)$,

which is true, as the logarithm function is increasing and

$8.4 \times 9.8 > 10$]

Thus the Answer is (c).

Q1/B

Any square number can be written as

$S = (10a + b)^2 = 100a^2 + 20ab + b^2$, where a & b are non-negative integers, and $0 \leq b \leq 9$.

The last digit of this number will be the same as the last digit of b^2. Considering the possible values of b, the possible last digits are: $1, 4, 9, 6, 5$ & 0.

So (b) can be ruled out.

Considering (e): b has to be 0;

Then either a is a multiple of 10, say $10M$, in which case

$S = 10000M^2$, which cannot equal 987,654,000;

or a isn't a multiple of 10, in which case $S = 100a^2$,

which also cannot equal 987,654,000

So (e) can be ruled out.

Considering (a): $99{,}999{,}999 = 10^8 - 1 = (10^4 + 1)(10^4 - 1)$.

Any integer square root of this number has to be smaller than $10^4 + 1$ and greater than $10^4 - 1$; ie it would have to be 10^4

So (a) can be ruled out.

[Alternatively: For a large number, 1 less than a square clearly cannot be a square itself.]

For both (c) and (d), we require

$S = (10a + 5)^2 = 100a^2 + 100a + 25$

For (d), this would mean that $100a^2 + 100a = 713{,}291{,}010$

And so (d) can be ruled out.

Thus the Answer is (c).

Q1/C

Let R & r be the radii of the large and small circles,

respectively. Then $\pi R^2 = 10\pi r^2$, so that $R^2 = 10r^2$.

Let O_L & O_S be the centres of the large and small circles,

respectively. Let S_L & S_S be the squares with diagonals

BO_L & $O_S D$, respectively. [The other corners of the two squares

will be where the circles meet the square $ABCD$, by symmetry.]

Then the diagonals of S_L & S_S are $\sqrt{2}\,R$ & $\sqrt{2}\,r$, respectively.

Hence, forming two expressions for the diagonal of square $ABCD$:

$\left(\sqrt{2}\,R + R\right) + \left(r + \sqrt{2}\,r\right) = \sqrt{2}$,

so that $(R + r)\left(1 + \sqrt{2}\right) = \sqrt{2}$,

and hence $R + r = \frac{\sqrt{2}}{1+\sqrt{2}} = \frac{\sqrt{2}(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)} = \frac{2-\sqrt{2}}{2-1} = 2 - \sqrt{2}$

[The information about the relative sizes of the two circles wasn't actually needed! The inclusion of unnecessary information is not in the spirit of most Maths questions, and is presumably an error on the question setter's part.]

Thus the Answer is (b).

Q1/D

The required solutions will be solutions of either

$((x^2-1)^2-2)^2-3=2$ or $((x^2-1)^2-2)^2-3=-2$;

ie $((x^2-1)^2-2)^2=5$ or $((x^2-1)^2-2)^2=1$,

and therefore of one of the following:

(a)$(x^2-1)^2=\sqrt{5}+2$ [$(x^2-1)^2=-\sqrt{5}+2<0$ has no sol'ns]

(b) $(x^2-1)^2=1+2=3$ or (c) $(x^2-1)^2=-1+2=1$

Then (a) $\Leftrightarrow x^2-1=\pm\sqrt{\sqrt{5}+2}$,

(b) $\Leftrightarrow x^2-1=\pm\sqrt{3}$ and (c) $\Leftrightarrow x^2-1=\pm 1$

And then (a) produces just 2 sol'ns (x_1 & $-x_1$),

as $1-\sqrt{\sqrt{5}+2}<0$

And (b) also produces 2 sol'ns, as $1-\sqrt{3}<0$

And finally, (c) produces 2 sol'ns from $x^2=2$,

and 1 sol'n from $x^2=0$.

So, in total there are 7 sol'ns.

[Note: Some mathematicians prefer to talk about 'the solution' of the equation being $x=x_1$ or $x=x_2$ ...]

Thus the Answer is (c).

Q1/E

There are $3^{10}-1$ positive whole numbers less than 3^{10}. The sum of these is $\frac{1}{2}(3^{10}-1)3^{10}$.

The sum of the powers of 3 is $1 + 3 + 3^2 + \cdots + 3^9 = \frac{3^{10}-1}{3-1}$

So the required sum is: $\frac{1}{2}(3^{10} - 1)3^{10} - \frac{1}{2}(3^{10} - 1)$

$= \frac{1}{2}(3^{10} - 1)^2$

Thus the Answer is (a).

Q1/F

$(1 - 2x)^5(1 + 4x^2)^5(1 + 2x)^5$

$= [(1 + 4x^2)(1 - 4x^2)]^5$

$= (1 - 16x^4)^5$

The coefficient of x^{12} is

$\binom{5}{3}(-16)^3 = -\binom{5}{2}2^{12} = -10 \times 2^{12} = -5 \times 2^{13}$

Thus the Answer is (a).

Q1/G

The existence of repeated roots for the given eq'ns means that

$b^2 - 4ac = 0$ & $c^2 - 4ba = 0$

We are now interested in $D = a^2 - 4cb$

We can use the 1st 2 eq'ns to obtain expressions for b & c in terms of a [in order to express D in terms of a, in view of option (e)]:

First of all, $b^2 - 4ac = 0 \Rightarrow b^4 = 16a^2c^2$

Then $c^2 - 4ba = 0 \Rightarrow \frac{b^4}{16a^2} = 4ba$

$\Rightarrow b^3 = 64a^3 \Rightarrow b = 4a$

Similarly, $c^2 - 4ba = 0 \Rightarrow c^4 = 16b^2a^2$

Then $b^2 - 4ac = 0 \Rightarrow \frac{c^4}{16a^2} = 4ac$

$\Rightarrow c^3 = 64a^3 \Rightarrow c = 4a$

Then $D = a^2 - 4cb = a^2 - 4(4a)(4a) = -63a^2$,

which means that the 3rd quadratic eq'n has no real roots.

Thus the Answer is (a).

[Hopefully no candidates will have selected (d)!]

Q1/H

Consider an isosceles triangle with sides $10, 10$ & $2a$.

The height of the triangle is $\sqrt{10^2 - a^2}$.

The area of the triangle is $\frac{1}{2}(2a)\sqrt{10^2 - a^2}$

Consider the square of this area: $a^2(100 - a^2)$

Writing $A = a^2$, and write $E = A(100 - A)$

As E is an n-shaped quadratic, E is maximised when $A = 50$,

and $2a = 2\sqrt{50} = 10\sqrt{2}$, which is just under 15

The answer is likely to be (d), but (c) has to be ruled out.

For (c), $a^2(100 - a^2) = 5^2(100 - 5^2) = 25 \times 75$

For (d), $a^2(100 - a^2) = (\frac{15}{2})^2\left(100 - (\frac{15}{2})^2\right)$

$= \frac{225}{16}(175) = 25 \times \frac{9 \times 175}{16}$

So we need to verify that $\frac{9\times175}{16} > 75$,

which is equivalent to $9 \times 175 - 16 \times 75 > 0$,

or $9 \times 7 - 16 \times 3 > 0$, which is the case.

Thus the Answer is (d).

Q1/I

We can write $p(x) = Ax(x - M)^2$, where A is a constant.

Then $1 = A(1 - M)^2$ and $2 = 2A(2 - M)^2$,

so that $\frac{1}{A} = (1 - M)^2 = (2 - M)^2$,

and hence $1 - 2M = 4 - 4M$,

and so $2M = 3$ and $M = \frac{3}{2}$

[The condition that $M > 0$ doesn't seem to have been needed!]

Thus the Answer is (e).

Q1/J

For $1 < x < 2, 0 < log_2 x < 1$ and so $\lfloor log_2 x \rfloor = 0$, and $f(x) = 1$

For $\frac{1}{2} < x < 1, -1 < log_2 x < 0$ and so $\lfloor log_2 x \rfloor = -1$,

and $f(x) = \frac{4}{3}$

For $\frac{1}{4} < x < \frac{1}{2}, -2 < log_2 x < -1$ and so $\lfloor log_2 x \rfloor = -2$,

and $f(x) = \left(\frac{4}{3}\right)^2$

The value of the required integral isn't changed by removing

$x = 1, 0, -1, -2, \ldots$ from the domain $0 < x < 2$.

So $\int_0^2 f(x)\, dx = (2-1)(1) + \left(1 - \frac{1}{2}\right)\left(\frac{4}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right)\left(\frac{4}{3}\right)^2 + \cdots$

$= 1 + \frac{1}{2}\left(\frac{4}{3}\right) + \left(\frac{1}{2}\right)^2\left(\frac{4}{3}\right)^2 + \cdots$

which is an infinite Geometric series with common ratio $\frac{1}{2}\left(\frac{4}{3}\right) = \frac{2}{3}$

So $\int_0^2 f(x)\, dx = \frac{1}{1-\frac{2}{3}} = 3$

Thus the Answer is (c).