Logarithms (MAT) – Ideas & Exercises (11 pages; 14/12/24)

Rewriting logarithm equation as exponential equation:

$log_a b = c \iff a^c = b$

Write $3 + 2log_2 5$ in the form $log_a b$

$3 + 2log_2 5 = 3log_2 2 + log_2(5^2)$

$= log_2(2^3) + log_2(5^2) = log_2(8 \times 25) = log_2(200)$

$log_a b \, log_b c = log_a c$ or $log_b c = \frac{log_a c}{log_a b}$

Prove this result

$log_a b\ log_b c = log_a c$ or $log_b c = \frac{log_a c}{log_a b}$

Solution

rtp $log_a b\ log_b c = log_a c$ (*)

Method 1

Let $b = a^x$ & $c = b^y$

Then $c = (a^x)^y = a^{xy}$

and $log_a c = xy = log_a b\ log_b c$, as required

Method 2

(*) is equivalent to $a^{log_a b\ log_b c} = a^{log_a c}$ (as $y = a^x$ is an increasing function)

ie $(a^{log_a b})^{log_b c} = c$ (**)

and the LHS equals $b^{log_b c} = c$, so that (**) holds, and hence (*) holds also

Relation connecting $log_x y$ and $log_y x$?

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$log_b c = \frac{log_a c}{log_a b}$

Writing $b = x, c = y$ & $a = y$: $log_x y = \frac{1}{log_y x}$

Demonstrate that $y = log_a x$ is a concave function when $a > 1$.

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First of all, $log_a x = log_a e \,.\, log_e x$ or $\frac{1}{log_e a} \,.\, lnx = \frac{lnx}{lna}$

Then $\frac{dy}{dx} = \frac{1}{xlna}$

and $\frac{d^2y}{dx^2} = -\frac{1}{x^2 lna} < 0$ (as $a > 1$);

ie the gradient is always decreasing.

In other words, $y = log_a x$ is a concave function.

Use linear interpolation to show that $log_8 36 > \frac{3}{2}$

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As $log_8 8 = 1$ and $log_8 64 = 2,$ and as $y = log_8 x$ is a concave function,

linear interpolation $\Rightarrow log_8\left[\frac{1}{2}(8 + 64)\right] > \frac{1}{2}(1 + 2)$

ie $log_8 36 > \frac{3}{2}$

Find an upper bound of the form $\frac{m}{n}$ for $log_2 3$

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Suppose that $log_2 3 < \frac{m}{n}$

Then $3 < 2^{(\frac{m}{n})}$ and $3^n < 2^m$

As $243 = 3^5 < 2^8 = 256, log_2 3 < \frac{8}{5}$

[and $\frac{8}{5}$ is a reasonably low upper bound, as 243 & 256 are reasonably close]

Given that $log_2 3 < \frac{8}{5}$, show that $log_2 12 < \frac{18}{5}$

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$log_2 12 = log_2(3 \times 4) = log_2 3 + log_2 4 < \frac{8}{5} + 2 = \frac{18}{5}$,

Show that $log_5 10 < \frac{3}{2}$

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$log_5 10 < \frac{3}{2} \Leftrightarrow 10 < 5^{\left(\frac{3}{2}\right)}$ (as the log function is increasing)

$\Leftrightarrow 10^2 < 5^3 \Leftrightarrow 100 < 125$

What relation is there between $log_2 a$ and $log_8 a$?

[Idea: Consider relation between powers.]

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Solution

Method 1

$log_2 a \times log_a 8 = log_2 8 = 3,$

so that $log_2 a = \frac{3}{log_a 8} = 3log_8 a$

[Standard result: $log_a b \times log_b a = log_a a = 1,$

so that $log_b a = \frac{1}{log_a b}$]

Method 2

$(2^3)^p = 2^{3p} = a$ (say)

so that $log_8 a = \frac{1}{3} log_2 a$

Show that $\log(4-\sqrt{15}) = -\log(4+\sqrt{15})$

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Solution

$\log(4-\sqrt{15}) = -log\left(\frac{1}{4-\sqrt{15}}\right) = -log\left(\frac{4+\sqrt{15}}{16-15}\right) = -\log\,(4+\sqrt{15})$

[or consider $\log(4-\sqrt{15}) + \log(4+\sqrt{15})$]