

Maclaurin Series - Exercises (Solutions) (8 pages; 19/1/21)

Easier

(1) Find a Maclaurin expansion (with 3 non-zero terms) for $\sin^2 x$ by two methods

Solution

Method 1

$$\begin{aligned}\sin^2 x &= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) \\ &= x^2 + \frac{x^6}{(3!)^2} + 2(x)\left(-\frac{x^3}{3!}\right) + 2x\left(\frac{x^5}{5!}\right) + \dots \\ &= x^2 - \frac{x^4}{3} + \left(\frac{1}{36} + \frac{1}{60}\right)x^6 + \dots\end{aligned}$$

[though in fact the next term is negative]

$$= x^2 - \frac{x^4}{3} + \frac{8}{180}x^6 + \dots = x^2 - \frac{x^4}{3} + \frac{2}{45}x^6 + \dots$$

Method 2

$$\begin{aligned}\sin^2 x &= \frac{1}{2}(1 - \cos 2x) \\ &= \frac{1}{2} - \frac{1}{2}\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \dots\right) \\ &= x^2 - \frac{x^4}{3} + \frac{2}{45}x^6 - \dots\end{aligned}$$

(2) Expand \sqrt{x} in powers of $x - 1$, and investigate the accuracy of the resulting approximation for $\sqrt{2}$ when 8 terms are taken.

Solution

$$\sqrt{x} = (1 + [x - 1])^{\frac{1}{2}} = 1 + \frac{1}{2}(x - 1) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}(x - 1)^2$$

$$+ \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}(x - 1)^3 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)}{4!}(x - 1)^4 + \dots$$

$$\text{Hence } \sqrt{2} \approx 1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} - \frac{5}{128} + \frac{7}{256} - \frac{21}{1024} + \frac{33}{2048} = 1.42139$$

(true figure: 1.41421)

Harder

(1) Find the 1st 3 non-zero terms of the Maclaurin expansions of the following functions, and the intervals of validity of the infinite series:

(i) $\ln(3 - 2x)$

(ii) $\ln\left(\frac{\sqrt{1+3x}}{1-2x}\right)$

(iii) $e^{\cos x}$

Solution

(i) $\ln(3 - 2x) = \ln 3 \left(1 - \frac{2x}{3}\right) = \ln 3 + \ln\left(1 - \frac{2x}{3}\right)$

$$= \ln 3 + \left(-\frac{2x}{3}\right) - \frac{\left(-\frac{2x}{3}\right)^2}{2} + \dots \approx \ln 3 - \frac{2x}{3} - \frac{2x^2}{9}$$

The infinite series is valid for $-1 < -\frac{2x}{3} \leq 1$; ie $-\frac{3}{2} \leq x < \frac{3}{2}$

[Note: Were we to go down the route of

$\ln(3 - 2x) = \ln(1 + [2 - 2x])$, there would be an infinite number of terms involving a constant, arising from the powers of $2 - 2x$]

$$\begin{aligned}
\text{(ii) } \ln\left(\frac{\sqrt{1+3x}}{1-2x}\right) &= \frac{1}{2}\ln(1+3x) - \ln(1-2x) \\
&= \frac{1}{2}\left(3x - \frac{(3x)^2}{2} + \frac{(3x)^3}{3} - \frac{(3x)^4}{4} + \dots\right) \\
&\quad - \left([-2x] - \frac{(-2x)^2}{2} + \frac{(-2x)^3}{3} - \frac{(-2x)^4}{4} + \dots\right) \\
&= \frac{7x}{2} + x^2\left(-\frac{9}{4} + 2\right) + x^3\left(\frac{9}{2} + \frac{8}{3}\right) + \dots \\
&= \frac{7x}{2} - \frac{x^2}{4} + \frac{43x^3}{6} + \dots
\end{aligned}$$

valid for x such that $-1 < 3x \leq 1$ and $-1 < -2x \leq 1$

$$\text{ie } -\frac{1}{3} < x \leq \frac{1}{3} \text{ and } \frac{1}{2} > x \geq -\frac{1}{2}$$

$$\text{ie } -\frac{1}{3} < x \leq \frac{1}{3}$$

$$\begin{aligned}
\text{(iii) } e^{\cos x} &= e^{1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots} \\
&= e\left\{1 + \left[-\frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right] + \frac{1}{2}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right)^2 + \dots\right\} \\
&= e\left\{1 - \frac{x^2}{2} + x^4\left(\frac{1}{24} + \frac{1}{8}\right) + \dots\right\} \\
&= e - \frac{ex^2}{2} + \frac{ex^4}{6} + \dots
\end{aligned}$$

valid for all x (as both $\cos x$ & e^x are valid for all values).

[Note that the expansion $e^{1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots}$
 $= 1 + \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right] + \frac{1}{2}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right)^2 + \dots$ would have involved an infinite number of constant terms (though their sum would be e).]

(2) Find the % errors (to 2sf) associated with the following Maclaurin approximations:

$$(i)(a) \cos x = 1 - \frac{x^2}{2!} \text{ for } x = \frac{\pi}{6} \text{ \& } x = \frac{\pi}{3}$$

$$(b) \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \text{ for } x = \frac{\pi}{6} \text{ \& } x = \frac{\pi}{3}$$

$$(ii)(a) e^x = 1 + x + \frac{x^2}{2!} \text{ for } x = 1 \text{ \& } x = 2$$

$$(b) e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \text{ for } x = 1 \text{ \& } x = 2$$

$$(iii) \ln(1+x) = x - \frac{1}{2}x^2 \text{ for } x = 0.1, x = 0.5 \text{ \& } x = 1$$

$$(b) \ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$$

$$\text{for } x = 0.1, x = 0.5 \text{ \& } x = 1$$

Solution

$$(i)(a) \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} = 0.86603 ; 1 - \frac{\left(\frac{\pi}{6}\right)^2}{2!} = 0.86292$$

$$\% \text{ error: } \left| \frac{0.86292 - 0.86603}{0.86603} \right| \times 100 = 0.36\%$$

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2} ; 1 - \frac{\left(\frac{\pi}{3}\right)^2}{2!} = 0.45169$$

$$\% \text{ error: } \left| \frac{0.45169 - 0.5}{0.5} \right| \times 100 = 9.7\%$$

$$(b) 1 - \frac{\left(\frac{\pi}{6}\right)^2}{2!} + \frac{\left(\frac{\pi}{6}\right)^4}{4!} = 0.86605$$

$$\% \text{ error: } \left| \frac{0.86605 - 0.86603}{0.86603} \right| \times 100 = 0.0023\%$$

$$1 - \frac{\left(\frac{\pi}{3}\right)^2}{2!} + \frac{\left(\frac{\pi}{3}\right)^4}{4!} = 0.50180$$

$$\% \text{ error: } \left| \frac{0.50180 - 0.5}{0.5} \right| \times 100 = 0.36\% \text{ (sic)}$$

$$(ii)(a) e^1 = 2.71828; 1 + 1 + \frac{1^2}{2!} = 2.5$$

$$\% \text{ error: } \left| \frac{2.5 - 2.71828}{2.71828} \right| \times 100 = 8.0\%$$

$$e^2 = 7.38906; 1 + 2 + \frac{2^2}{2!} = 5$$

$$\% \text{ error: } \left| \frac{5 - 7.38906}{7.38906} \right| \times 100 = 32\%$$

$$(b) 1 + 1 + \frac{1^2}{2!} + \frac{1^3}{3!} + \frac{1^4}{4!} = 2.70833$$

$$\% \text{ error: } \left| \frac{2.70833 - 2.71828}{2.71828} \right| \times 100 = 0.37\%$$

$$1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} = 7$$

$$\% \text{ error: } \left| \frac{7 - 7.38906}{7.38906} \right| \times 100 = 5.3\%$$

$$(iii)(a) \ln(1 + 0.1) = 0.095310; 0.1 - \frac{1}{2} (0.1)^2 = 0.095$$

$$\% \text{ error: } \left| \frac{0.095 - 0.095310}{0.095310} \right| \times 100 = 0.33\%$$

$$\ln(1 + 0.5) = 0.40547; 0.5 - \frac{1}{2} (0.5)^2 = 0.375$$

$$\% \text{ error: } \left| \frac{0.375 - 0.40547}{0.40547} \right| \times 100 = 7.5\%$$

$$\ln(1 + 1) = 0.69315; 1 - \frac{1}{2} (1)^2 = 0.5$$

$$\% \text{ error: } \left| \frac{0.5 - 0.69315}{0.69315} \right| \times 100 = 28\%$$

$$(b) 0.1 - \frac{1}{2} (0.1)^2 + \frac{1}{3} (0.1)^3 - \frac{1}{4} (0.1)^4 = 0.095308$$

$$\% \text{ error: } \left| \frac{0.095308 - 0.095310}{0.095310} \right| \times 100 = 0.0021\%$$

$$0.5 - \frac{1}{2} (0.5)^2 + \frac{1}{3} (0.5)^3 - \frac{1}{4} (0.5)^4 = 0.40104$$

$$\% \text{ error: } \left| \frac{0.40104 - 0.40547}{0.40547} \right| \times 100 = 1.1\%$$

$$1 - \frac{1}{2}(1)^2 + \frac{1}{3}(1)^3 - \frac{1}{4}(1)^4 = 0.58333$$

$$\% \text{ error: } \left| \frac{0.58333 - 0.69315}{0.69315} \right| \times 100 = 16\%$$

[Note that we are at the extreme end of the range of validity.]

(3) Use the 1st 5 terms of a Maclaurin expansion to find an approximate value for $P(Z < 1)$, where $Z \sim N(0,1)$ and Z has pdf

$$f(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right)$$

Solution

$$P(Z < 1) = 0.5 + \int_0^1 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right) dz$$

$$\approx 0.5 + \frac{1}{\sqrt{2\pi}} \int_0^1 \left[1 + \left(-\frac{1}{2}z^2\right) + \frac{\left(-\frac{1}{2}z^2\right)^2}{2!} + \frac{\left(-\frac{1}{2}z^2\right)^3}{3!} + \frac{\left(-\frac{1}{2}z^2\right)^4}{4!} \right] dz$$

$$= 0.5 + \frac{1}{\sqrt{2\pi}} \left[z - \frac{z^3}{6} + \frac{z^5}{40} - \frac{z^7}{336} + \frac{z^9}{3456} \right]_0^1$$

$$= 0.5 + \frac{1}{\sqrt{2\pi}} \left(1 - \frac{1}{6} + \frac{1}{40} - \frac{1}{336} + \frac{1}{3456} \right)$$

$$= 0.5 + \frac{1}{\sqrt{2\pi}} (0.8556465) = 0.84135$$

[Normal tables give 0.8413]

(4) Use 3 terms of a Maclaurin expansion of $\ln\left(\frac{1+x}{1-x}\right)$ to find an approximate value for $\ln\left(\frac{2}{3}\right)$

Solution

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$

$$\begin{aligned}
&= \left\{ x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots \right\} \\
&- \left\{ [-x] - \frac{[-x]^2}{2} + \frac{[-x]^3}{3} - \frac{[-x]^4}{4} + \frac{[-x]^5}{5} - \dots \right\} \\
&= 2 \left\{ x + \frac{x^3}{3} + \frac{x^5}{5} + \dots \right\}
\end{aligned}$$

(valid, provided that $-1 < x \leq 1$ and $-1 < -x \leq 1$;

ie $-1 < x \leq 1$ and $1 > x \geq -1$

ie $-1 < x < 1$)

Suppose that $\frac{1+x}{1-x} = \frac{2}{3}$

Then $3 + 3x = 2 - 2x$, so that $5x = -1$ and $x = -\frac{1}{5}$

(and this is within the limits of validity).

$$\text{So } \ln\left(\frac{2}{3}\right) \approx 2 \left\{ \left[-\frac{1}{5}\right] + \frac{\left[-\frac{1}{5}\right]^3}{3} + \frac{\left[-\frac{1}{5}\right]^5}{5} \right\} = -0.40546 = -0.405 \text{ (3sf)}$$

[The true value of $\ln\left(\frac{2}{3}\right)$ is -0.40547 (5sf). Note that $x = -\frac{1}{5}$ is closer to the value of 0 (about which the Maclaurin expansion is centred) than $x = \frac{1}{3}$ [giving $\ln\left(1 - \frac{1}{3}\right)$], so that greater accuracy is to be expected.]

(5) Find the first 3 non-zero terms, as well as the general term in the Maclaurin expansion of $\cosh^3 x$

Solution

$$\text{Let } f(x) = \cosh^3 x \quad [f(0) = 1]$$

$$\text{Then } f'(x) = 3\cosh^2 x \sinh x \quad [f'(0) = 0]$$

$$\text{and } f''(x) = 6\cosh x \sinh^2 x + 3\cosh^3 x \quad [f''(0) = 3]$$

$$= 6\cosh x(\cosh^2 x - 1) + 3\cosh^3 x = 9\cosh^3 x - 6\cosh x$$

$$= 9f(x) - 6\cosh x$$

Then $f'''(x) = 9f'(x) - 6\sinh x$ [$f'''(0) = 0$]

and $f^{(4)}(x) = 9f''(x) - 6\cosh x$

$$= 81f(x) - 6\cosh x(1 + 9) \quad [f^{(4)}(0) = 81 - 60 = 21]$$

Clearly $f^{(2r+1)}(0) = 0$ and hence $a_{2r+1} = 0$

$$f^{(6)}(x) = 81f''(x) - 6\cosh x(1 + 9)$$

$$= 81(9)f(x) - 6\cosh x(1 + 9 + 81)$$

Thus $f^{(2r)}(x) = 3^{2r}f(x) - 6\cosh x(1 + 9 + \dots + 9^{r-1})$

and so $a_{2r} = \frac{f^{(2r)}(0)}{(2r)!} = \frac{1}{(2r)!} \left\{ 3^{2r} - 6 \frac{(9^r - 1)}{(9 - 1)} \right\}$

$$= \frac{1}{(2r)!} \left\{ 3^{2r} - 3 \frac{(3^{2r} - 1)}{4} \right\}$$

$$= \frac{3^{2r} + 3}{4(2r)!}$$

So $\cosh^3 x = 1 + \frac{3x^2}{2!} + \frac{21x^4}{4!} + \dots + \frac{(3^{2r} + 3)x^{2r}}{4(2r)!} + \dots$

$$= 1 + \frac{3x^2}{2} + \frac{7x^4}{8} + \dots + \frac{(3^{2r} + 3)x^{2r}}{4(2r)!} + \dots$$

Note: In other cases, such as $\cosh^2 x$, if only required to find the first few terms of the expansion, it may well be easier to expand

$$\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!}\right) \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!}\right)$$