Notes on Linear Programming (5 pages; 21/5/24)

[These notes assume that the reader is already familiar with the basic ideas behind Linear Programming – see separate notes for this. In its most general sense, Linear Programming covers the Simplex method as well (see separate notes), but here we will only consider a graphical approach.]

(1) The graphical approach can only cope with 2 variables (usually x & y). However, if there is a constraint such as $x + 2y - z = 4$, then this can be used to eliminate z, and convert the problem to a 2-variable problem.
It will usually be necessary to allow for a constraint such as $z \geq 0$, and this will become $x + 2y - 4 \geq 0$

(2) If only integer solutions are acceptable, then there are several possible approaches:
(i) Compare the value of P at the (integer-valued) vertices of the surrounding square (provided they lie within the feasible region).
(ii) Consider the integer values of x lying on either side of the (non-integer) solution for x. For each of these, find the integer value of y that maximises P, whilst satisfying the constraints. (Because x is being fixed, this can be done algebraically.)
(iii) The above two approaches are not guaranteed to find the

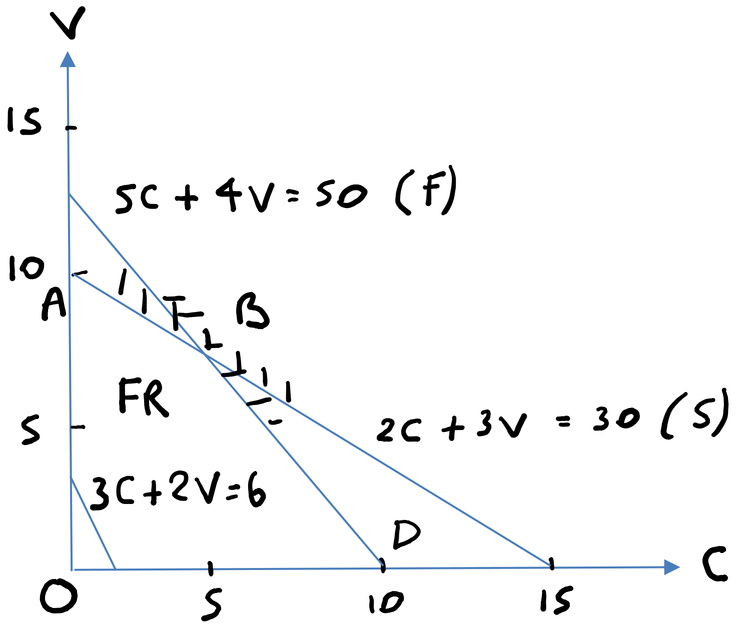
optimal integer-valued solution. The ‘Branch and Bound’ method will find an optimal solution, but is more involved.

(3) A complication arises when the Objective function (in the form $y = 2x + P$, for example) is parallel to a constraint line. In this case, a potential solution exists for all points on the line segment between two vertices of the feasible region.

(4) To deal with a constraint of the form $x + y < 4$, for example: Use $x + y \leq 4$ instead, and reduce x *or* y slightly, if necessary.

(5) If x (for example) can be negative, then replace x with $x_1 - x_2$, where $x_1, x_2 \geq 0$. This allows x to be negative, if necessary.

(6) If the gradients of the Objective line and one or more of the constraint lines are similar, then it may be difficult to see (by eye) which vertex of the Feasible Region is furthest away from the Origin, when the Objective line is moved (when P is being maximised). In this case, it can help to compare the gradients themselves, as below:



Referring to the diagram, the objective line $P = 3C + 2V$ (or $V = \frac{P}{2} - \frac{3C}{2}$) will be parallel to $3C + 2V = 6$, and needs to be as far away from O as possible, in order to maximise P.

As the objective line is moved away from O, the required vertex will be the one that the objective line crosses as it leaves the Feasible Region. In this example it is D. This can be determined from the gradients, as follows:

(F) $5C + 4V = 50$: gradient is $-\frac{5}{4}$

(S) $2C + 3V = 30$: gradient is $-\frac{2}{3}$

(P) $P = 3C + 2V$: gradient is $-\frac{3}{2}$

Line S is the 1st constraint line that P comes into contact with.

P has a steeper gradient than S, so that P first comes into contact with S at the top end, and so the vertex B will be further away from O than A, as P is moved away from O.

So B is preferred to S.

Beyond B, F is the critical constraint line.

P has a steeper gradient than F, so that vertex D will be further away from O than B, and therefore preferred to B.

Thus, D is the optimal vertex.

[If the gradients are not too similar, the above approach can be performed by inspection of the sketch, by comparing the objective line with a constraint line, and seeing at which end they converge. The required vertex will then lie toward the end opposite the point of convergence.]

(7) Example of a more complicated constraint

A company manufactures 3 liquid products: X, Y and Z, sold in drums. There is enough of a constituent chemical to make 45 drums of X, or 60 drums of Y, or 90 drums of Z. Formulate this constraint as an inequality.

Solution

If only X is being produced, then the constraint can be written as

$\frac{X}{45} \leq 1$

If (say) up to half of X's quota is to be used, and up to half of Y's, then the constraint would be $\frac{X}{45} \leq 0.5$ & $\frac{Y}{60} \leq 0.5$;

or if X is favoured over Y, eg $\frac{X}{45} \leq 0.7$ & $\frac{Y}{60} \leq 0.3$

The most flexible constraint involving all 3 products would be

$\frac{X}{45} + \frac{Y}{60} + \frac{Z}{90} \leq 1$