# Notes on the Simplex Method (5 pages; 20/5/24)

[These notes assume that the reader is already familiar with the basic ideas behind the Simplex method – see separate notes for this.]

(1) Terminology

(i) standard form is used to describe situations where:

- the objective function is to be maximised (ie rather than minimised)

- the constraints are all of the $\leq$ form (except for $x \geq 0$ etc)

(ii) augmented form: where the inequalities representing the constraints have been converted to equations involving slack variables $s_i$ (in the case of $\leq$ constraints) or surplus variables $s_i$ (in the case of $\geq$ constraints, where either the Two-Stage Simplex or the Big M method is used).

(iii) After each pass of the method, variables that only appear in one row of the tableau, and have a coefficient of 1, are referred to as basic variables. Their value at this stage of the method will be the right-hand value for the row in question. This is possible because the other (non-basic) variables (that appear in more than one row) are being set equal to zero.

(2) The idea behind the Simplex method is that the Objective

equation (involving $P$) is to be manipulated in such a way that $P$ can attain a higher value. This is done by maximising the value of one of the variables $x, y, \dots, s_1, s_2, \dots$ with a negative coefficient; eg, with $P - x - 2y + z - 3s_1 + s_2 - s_3 = 7$, it is conventional to maximise $s_1$, because (all else being equal) this gives the best chance of maximising $P$, when $x, y, z, s_2$ & $s_3$ are set equal to zero (and then $P$ becomes $7 + 3s_1$).

(3) Although it is, in theory, possible to minimise $P$ by maximising the value of a variable with a **positive** coefficient, it is conventional to use the Simplex method only to maximise $P$ (but see discussion below about the Two-Stage Simplex method). If $P$ is to be minimised, then we can instead maximise $P' = -P$.

(4) When carrying out the Ratio test, we need only consider coefficients in the pivot column that have a positive coefficient. This assumes that the right-hand values of the constraint rows are non-negative (which is the convention).
For example, when attempting to maximise $x$ in the constraint row $-2x + 3y + s_1 + s_2 = 12$, there is no upper limit to the possible value of $x$, as another slack variable ($s_3$ say) could always be created, so that $-2x + 3y + s_1 + s_2 + s_3 = 12$, with $x$ being large, $y = s_1 = s_2 = 0$, and $s_3$ being large. Thus, this

particular constraint row places no restriction on the size of $x$, and therefore doesn't need to be considered when carrying out the Ratio test.

(5) After each pass of the Simplex method, the solution so far will correspond to a vertex of the feasible region - in the case of a graphical solution where there are only two variables (usually $x$ & $y$). The initial position (where $x = 0, y = 0$) corresponds to the Origin.

[Note that, having maximised one of the variables $x, y, \dots, s_1, s_2, \dots$, there is nothing to stop that variable being reduced in a subsequent pass of the method. (Consider what happens when we move from one vertex of the feasible region to another.)]

(6) Two-Stage Simplex

A problem occurs where there are $\geq$ constraints, because the usual starting point for the Simplex method - where $x = 0, y = 0$ etc – cannot be applied (eg if $x + 2y - s_1 = 4$, then setting $x = 0, y = 0$ would mean that $s_1$ would have to equal $-4$; but the surplus variable $s_1$ is required to be non-negative). We get round this by introducing an artificial variable; $x + 2y - s_1 + a_1 = 4$ for the above example. This allows an initial solution of $x = 0, y = 0, s_1 = 0, a_1 = 4$, and we hope to be able to minimise

$a_1$, so that - in the process - $x$ & $y$ move into the feasible region.

(7) Rather illogically, for the 1st stage of the Two-Stage Simplex method, where $A\ (= a_1 + a_2 + \cdots)$ has to be minimised, it is conventional to achieve this by maximising a variable that has a **positive** coefficient (whereas, when it comes to minimising $P$, the convention is to maximise $-P$, and then maximise a variable that has a negative coefficient).

(8) In order to be able to apply the Simplex method, the right-hand sides of the constraint rows need to be non-negative. (Only the Objective row may have a negative value on the right-hand side.)
A constraint such as $3x + 2y - z \geq -2$ can be rewritten as $-3x - 2y + z \leq 2$ (which also has the advantage that it is a standard $\leq$ inequality). However, a constraint such as $3x + 2y - z \leq -2$ would have to be rewritten as $-3x - 2y + z \geq 2$, and then the Two-stage Simplex method would need to be employed.

(9) Constraints that are equalities (eg $x - 2y = 5$) can be dealt with by replacing them with two inequalities (eg $x - 2y \leq 5$ and $x - 2y \geq 5$. The Two-stage Simplex method is then needed, in

order to be able to cope with the $\geq$ inequality.