Rational Functions (5 pages; 12/7/24)

(1) Example A: Sketch the curve $y = \frac{x^2+1}{x^2-3x}$, noting any stationary points.

(i) There are no solutions when $x = 0$ or $y = 0$, so the curve doesn't cross the x or y axes.

(ii) There are vertical asymptotes where $x^2 - 3x = 0$;

ie at $x = 0$ and $x = 3$

(iii) $\frac{x^2+1}{x^2-3x} = \frac{1+1/x^2}{1-3/x} \rightarrow \frac{1}{1} = 1$ as $x \rightarrow \pm\infty$;

ie the horizontal asymptote is $y = 1$

[Note that $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to\infty} f(x)}{\lim\limits_{x\to\infty} g(x)}$ only when $\lim\limits_{x\to\infty} f(x)$ & $\lim\limits_{x\to\infty} g(x)$ are constants. [See "A course of Pure Mathematics" by G.H. Hardy (CUP 1933): Theorem IV of section 66]

Example where $\frac{\lim f(x)}{\lim g(x)} \neq lim \frac{f(x)}{g(x)}$:

$y = \frac{6x^2-5x+3}{3x-1} = \cdots = 2x - 1 + \frac{2}{3x-1} \rightarrow 2x - 1$

[taking $f(x) = \frac{2}{3x-1}$ and $g(x) = 1$, and noting that $\lim\limits_{x\to\infty} f(x) = 0$]

but $\frac{6x-5+3/x}{3-1/x}$ wrongly suggests a limit of $\frac{6x-5}{3} = 2x - \frac{5}{3}$]

(iv) Behaviour at the vertical asymptotes

Writing $y = \frac{x^2+1}{x(x-3)}$,

$x = -\delta \Rightarrow \frac{+}{(-)(-)} > 0$ & $x = \delta \Rightarrow \frac{+}{(+)(-)} < 0$

and $x = 3 - \delta \Rightarrow \frac{+}{+(-)} < 0$ & $x = 3 + \delta \Rightarrow \frac{+}{+(+)} > 0$

(v) Behaviour at the horizontal asymptote

For large x, say $x = 100$: $y = 1.03$

For large negative x, say $x = -100$: $y = 0.97$

Thus the curve approaches $y = 1$ from above as $x \to \infty$ and from below as $x \to -\infty$

(vi) To find the stationary points, consider the values of k for which solutions exist for $\frac{x^2+1}{x^2-3x} = k$;

ie $x^2(1-k) + 3kx + 1 = 0$ (*)

Solutions exist when $(3k)^2 - 4(1-k) \geq 0$

ie $9k^2 + 4k - 4 \geq 0$

The roots of $9k^2 + 4k - 4 = 0$ are $k = \frac{-4 \pm \sqrt{16+144}}{18} = \frac{-2 \pm 2\sqrt{10}}{9}$

$= 0.48051$ & -0.92495

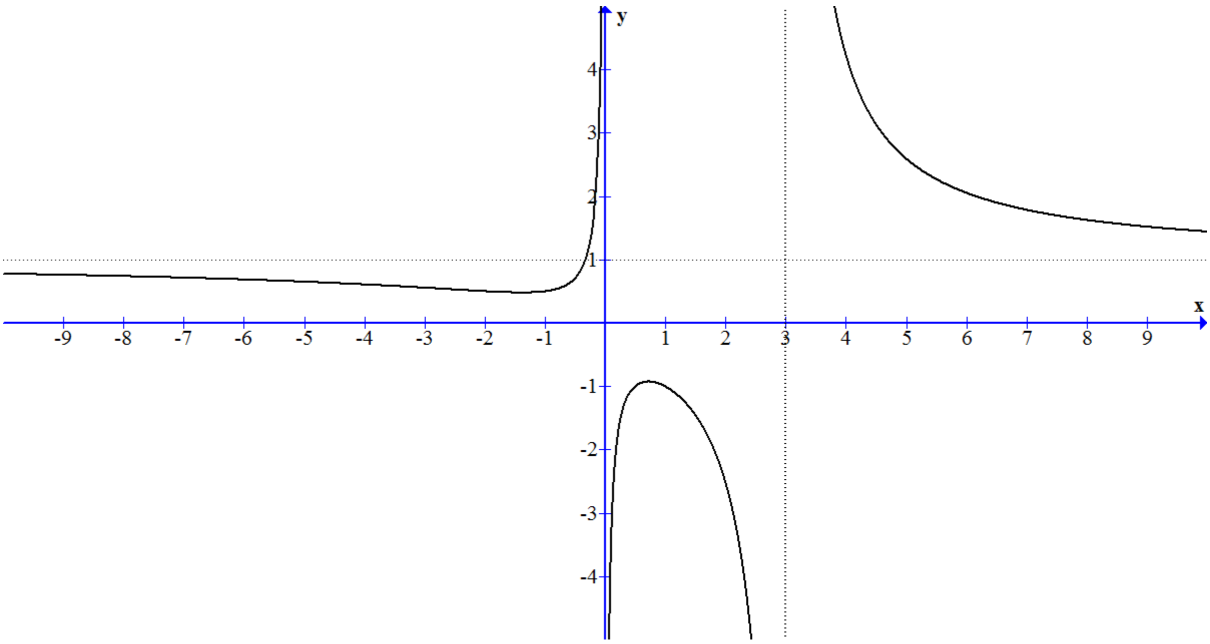
Thus the curve exists for values of k outside the range -0.92495 to 0.48051

(vii) To find the x coordinates of the stationary points, from (*):

$x = \frac{-3k}{2(1-k)} = 0.72076$ & -1.38745

Thus the stationary points are $(-1.39, 0.48)$ & $(0.72, -0.92)$

(viii) Graph:



(2) Example B: Sketch the graph of $y = \frac{(x-1)(x+1)^2}{x^2(2x-3)}$

(i) $y = \frac{(x-1)(x+1)^2}{x^2(2x-3)} = 0$ when $x = 1$ $or - 1$ (repeated root)

(ii) Vertical asymptotes: $x = 0$ & $x = \frac{3}{2}$

(iii) Horizontal asymptote: $\lim_{x\to\infty} \frac{(x-1)(x+1)^2}{x^2(2x-3)} = \lim_{x\to\infty} \frac{(x^2-1)(x+1)}{2x^3-3x^2}$

$= \lim_{x\to\infty} \frac{x^3+x^2-x-1}{2x^3-3x^2} = \lim_{x\to\infty} \frac{1+\frac{1}{x}-\frac{1}{x^2}-\frac{1}{x^3}}{2-\frac{3}{x}} = \frac{1}{2}$; ie $y = \frac{1}{2}$

(iv) Behaviour at the vertical asymptotes

As $y = \frac{(x-1)(x+1)^2}{x^2(2x-3)}$, $x = \frac{3}{2} + \delta \Rightarrow y\ is\ \frac{(+)(+)}{(+)(+)} = (+)$

For $x = \frac{3}{2} - \delta$, only the sign of $2x - 3$ is going to change, so that

$y\ is\ (-)$.

However, note that, because of the x^2 term in $\frac{(x-1)(x+1)^2}{x^2(2x-3)}$,

$x = 0 + \delta$ and $x = 0 - \delta$ are both $\frac{(-)(+)}{(+)(-)} = (+)$

(v) Behaviour at the horizontal asymptote

When $x = 100, y = \frac{99(101)^2}{100^2(197)} = 0.513 > \frac{1}{2}$

When $x = -100, y = \frac{(-101)(-99)^2}{100^2(-203)} = 0.488 < \frac{1}{2}$

(vi) It is possible to investigate when the graph crosses

$y = \frac{1}{2}$, as follows:

$\frac{(x-1)(x+1)^2}{x^2(2x-3)} = \frac{1}{2} \Rightarrow 2(x-1)(x^2+2x+1) = 2x^3 - 3x^2$ (*)

$\Rightarrow 2x^3 + x^2(4-2) + x(2-4) - 2 = 2x^3 - 3x^2$

$\Rightarrow 5x^2 - 2x - 2 = 0$

$\Rightarrow x = \frac{2 \pm \sqrt{4-4(5)(-2)}}{10} = \frac{2 \pm \sqrt{44}}{10} = \frac{1 \pm \sqrt{11}}{5}$

ie the curve crosses $y = \frac{1}{2}$ in the interval $(0, \frac{1+\sqrt{16}}{5})$; ie $(0, 1)$

and in the interval $(\frac{1-\sqrt{16}}{5}, 0)$; ie $(-\frac{3}{5}, 0)$

and to the left of $\frac{1-3}{5} = -2/5$

[Note: The fact that $y = \frac{1}{2}$ is the horizontal asymptote ensures that the terms in x^3 cancel from both sides of (*), but had there been any terms in x^4 we would have been left with a cubic (ie generally not solvable by simple methods).]

(vii) Graph: