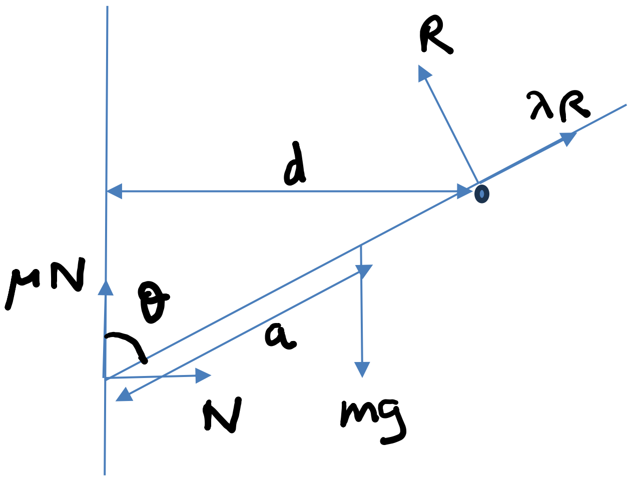
STEP 2016, P1, Q9 - Solution (5 pages; 3/5/24)

1st Part

The following force diagram applies for the rod:



It is clear* that, were the wall and the rail to be smooth, then the rod would slide down the wall, so that the frictional forces act as shown - opposing attempted motion.

*Alternatively, we can take moments about P to show that there would be a net anti-clockwise moment, were there to be no frictional forces.

N2L along the rod: $Nsin\theta + \lambda R + \mu Ncos\theta = mgcos\theta$ (1)

Perp. to the rod: $R + \mu Nsin\theta = Ncos\theta + mgsin\theta$ (2)

Taking moments about A: $mgsin\theta . a = R . \frac{d}{sin\theta}$ (3)

[Note that we are introducing the mass of the rod, but that the problem will not be dependent on it.]

Using (3) to eliminate mg (or $mgsin\theta$),

(1) becomes $Nsin\theta + \lambda R + \mu Ncos\theta = (\frac{Rd}{asin^2\theta})cos\theta \quad (1')$

and (2) becomes $R + \mu N sin\theta = N cos\theta + \frac{Rd}{asin\theta}$ $(2')$

We can then use $(1')$ & $(2')$ to equate two expressions for $\frac{N}{R}$:

From $(1')$, $\frac{N}{R}sin\theta + \lambda + \mu\frac{N}{R}cos\theta = \frac{dcos\theta}{asin^2\theta}$,

so that $\frac{N}{R} = \frac{\frac{dcos\theta}{asin^2\theta} - \lambda}{sin\theta + \mu cos\theta}$,

and from $(2')$, $1 + \mu\frac{N}{R}sin\theta = \frac{N}{R}cos\theta + \frac{d}{asin\theta}$,

so that $\frac{N}{R} = \frac{\frac{d}{asin\theta} - 1}{\mu sin\theta - cos\theta}$

Hence $\frac{\frac{dcos\theta}{asin^2\theta} - \lambda}{sin\theta + \mu cos\theta} = \frac{\frac{d}{asin\theta} - 1}{\mu sin\theta - cos\theta}$, so that

$(dcos\theta - \lambda asin^2\theta)(\mu sin\theta - cos\theta)$

$= (dsin\theta - asin^2\theta)(sin\theta + \mu cos\theta)$

Then, as the term $d\mu cos\theta sin\theta$ cancels from both sides,

$-dcos^2\theta - \lambda a\mu sin^3\theta + \lambda asin^2\theta cos\theta$

$= dsin^2\theta - asin^3\theta - a\mu sin^2\theta cos\theta$,

and so $d = asin^3\theta + a\mu sin^2\theta cos\theta - \lambda a\mu sin^3\theta + \lambda asin^2\theta cos\theta$,

and hence $dcosec^2\theta = asin\theta + a\mu cos\theta - \lambda a\mu sin\theta + \lambda acos\theta$

$= a(cos\theta(\mu + \lambda) + sin\theta(1 - \lambda\mu))$, which gives the required relation

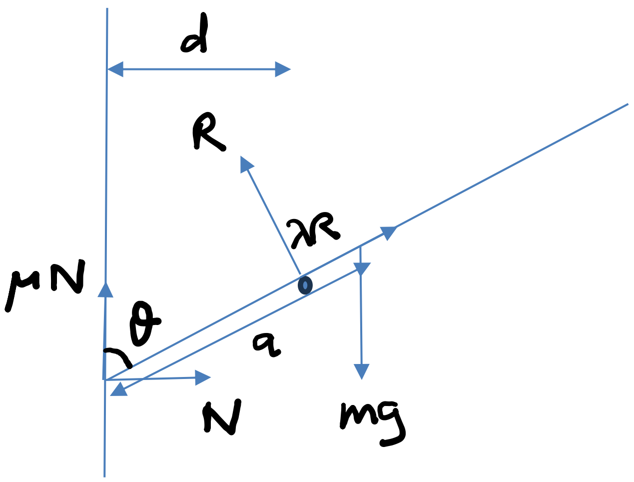
2nd Part

Clearly, in the extreme case of a very small value for d, it would not be possible to prevent the rod from rotating clockwise. For a larger value of d, it is not clear (without doing any calculations) whether the rod would tend to rotate clockwise or anti-clockwise, in the absence of frictional forces. However, the question tells us that there is limiting equilibrium, and since d is now taking a smaller value than in the 1st Part, this limiting equilibrium can only relate to a situation whether the frictional forces act down the wall and down the rod.

[In fact, as shown at the end, the rod has a tendency to rotate clockwise (in the absence of frictional forces) only when

$d < asin^3\theta$ (rather than $asin\theta$), although of course

$asin^3\theta < asin\theta$. So the question is arguably a bit misleading in implying that $d = asin\theta$ is some sort of critical point.]



N2L along the rod: $Nsin\theta - \lambda R - \mu Ncos\theta = mgcos\theta$ (1)

Perp. to the rod: $R - \mu Nsin\theta = Ncos\theta + mgsin\theta$ (2)

Taking moments about A: $mgsin\theta . a = R . \frac{d}{sin\theta}$ (3)

Using (3) to eliminate mg (or $mgsin\theta$),

(1) becomes $Nsin\theta - \lambda R - \mu Ncos\theta = (\frac{Rd}{asin^2\theta})cos\theta$ (1′)

and (2) becomes $R - \mu Nsin\theta = Ncos\theta + \frac{Rd}{asin\theta}$ (2′)

We can then use $(1')$ & $(2')$ to equate two expressions for $\frac{N}{R}$:

From $(1')$, $\frac{N}{R}sin\theta - \lambda - \mu\frac{N}{R}cos\theta = \frac{dcos\theta}{asin^2\theta}$,

so that $\frac{N}{R} = \frac{\frac{dcos\theta}{asin^2\theta}+\lambda}{sin\theta - \mu cos\theta}$,

and from $(2')$, $1 - \mu\frac{N}{R}sin\theta = \frac{N}{R}cos\theta + \frac{d}{asin\theta}$,

so that $\frac{N}{R} = \frac{1-\frac{d}{asin\theta}}{\mu sin\theta + cos\theta}$

Hence $\frac{\frac{dcos\theta}{asin^2\theta}+\lambda}{sin\theta - \mu cos\theta} = \frac{1-\frac{d}{asin\theta}}{\mu sin\theta + cos\theta}$, so that

$(dcos\theta + \lambda asin^2\theta)(\mu sin\theta + cos\theta)$

$= (asin^2\theta - dsin\theta)(sin\theta - \mu cos\theta)$

Then, as the term $d\mu cos\theta sin\theta$ cancels from both sides,

$dcos^2\theta + \lambda a\mu sin^3\theta + \lambda asin^2\theta cos\theta$

$= asin^3\theta - a\mu sin^2\theta cos\theta - dsin^2\theta$,

and so $d = asin^3\theta - a\mu sin^2\theta cos\theta - \lambda a\mu sin^3\theta - \lambda asin^2\theta cos\theta$,

and hence $dcosec^2\theta = asin\theta - a\mu cos\theta - \lambda a\mu sin\theta - \lambda acos\theta$

$= a(sin\theta(1-\lambda\mu) - cos\theta(\mu+\lambda))$

Investigation into the critical point when the rod first has a tendency to rotate clockwise (in the absence of frictional forces) [for the 2nd Part]

From (1), (2) & (3) [of the 1st Part] with $\lambda = \mu = 0$,

$Nsin\theta = mgcos\theta$, $R = Ncos\theta + mgsin\theta$

and $M(A)_{cw} = mgsin\theta . a - R . \frac{d}{sin\theta}$

So $R = (mgcot\theta)cos\theta + mgsin\theta = \frac{mg}{sin\theta}(cos^2\theta + sin^2\theta)$

$= \frac{mg}{sin\theta}$, and hence $M(A)_{cw} = mg(asin\theta - \frac{d}{sin^2\theta})$

Thus the critical point when $M(A)_{cw} = 0$ is

$asin\theta - \frac{d}{sin^2\theta} = 0$; ie $d = asin^3\theta$