STEP 2018, P3, Q5(i) - Solution (5 pages; 14/5/24)

(i) $(k+1)(A_{k+1}-G_{k+1}) \geq k(A_k - G_k)$

$\Leftrightarrow (k+1)A_{k+1} - kA_k \geq (k+1)G_{k+1} - kG_k$

$\Leftrightarrow\ a_{k+1} \geq (k+1)G_{k+1} - kG_k$ (from the definition of A_n)

$\Leftrightarrow \frac{a_{k+1}}{G_k} \geq (k+1)\frac{G_{k+1}}{G_k} - k$ (as $G_k > 0$)

$\Leftrightarrow {\lambda_k}^{k+1} - (k+1)\theta_k + k \geq 0$, where $\theta_k = \frac{G_{k+1}}{G_k}$

So, in order to prove the required result, we need to show that

$\theta_k = \lambda_k$.

Now $\theta_k = \lambda_k \Leftrightarrow {\theta_k}^{k+1} = {\lambda_k}^{k+1}$, as both θ_k & λ_k are positive

$\Leftrightarrow \frac{{G_{k+1}}^{k+1}}{{G_k}^{k+1}} = \frac{a_{k+1}}{G_k}$ (*)

Now ${G_{k+1}}^{k+1} = a_1 \dots a_{k+1} = {G_k}^k a_{k+1}$,

so that LHS of (*) $= \frac{{G_{k+1}}^{k+1}}{{G_k}^{k+1}} = \frac{{G_k}^k a_{k+1}}{{G_k}^k . G_k} = \frac{a_{k+1}}{G_k} = RHS$ of (*), as required.

(ii) **1st Part**

The problem is equivalent to establishing that the graph

of $y = x^{k+1} + k$ lies on or above that of $y = (k+1)x$

for $x > 0$. [Sketches of these graphs can be visualised. When $x = 1$, both of these functions have the value $k+1$.]

[As we expect that $f(1) = 0$, $f(x)$ can be factorised:]

Now, $f(x) = x^{k+1} - (k+1)x + k$

$= (x-1)(x^k + x^{k-1} + \cdots + x - k)$

When $x = 1$, $f(x) = 0$.

When $0 < x < 1$, each $x^r < 1$, and so $x^k + x^{k-1} + \cdots + x < k$,

and hence $f(x) = -ve \ \times \ -ve$, and so $f(x) > 0$

When $x > 1$, each $x^r > 1$, and so $x^k + x^{k-1} + \cdots + x > k$,

and hence $f(x) = +ve \ \times \ +ve$, and so $f(x) > 0$ again.

Thus, $f(x) \geq 0$ when $x > 0$.

2nd Part

As $f(x) < 0$ when $0 < x < 1$, and $f(x) > 0$ when $x > 1$,

$f(x) = 0$ if and only if $x = 1$ (when $x > 0$).

($f(1) = 0$ was established in the 1st Part.)

[The official mark scheme establishes that there is a single stationary point at $x = 1$. It may be worth adding that the function is continuous (which then ensures that $f(x)$ cannot fall below $f(1)$).]

(iii)(a) With $x = \lambda_k = \left(\frac{a_{k+1}}{G_k}\right)^{\frac{1}{k+1}}$, $x > 0$, and so (from (ii))

${\lambda_k}^{k+1} - (k+1)\lambda_k + k \geq 0$

Then, from (i), $(k+1)(A_{k+1} - G_{k+1}) \geq k(A_k - G_k)$ (*)

As $A_1 = G_1 (= a_1)$, it therefore follows that

$2(A_2 - G_2) \geq 0$, and hence $A_2 - G_2 \geq 0$.

Then $3(A_3 - G_3) \geq 2(A_2 - G_2) \geq 0$, so that $A_3 - G_3 \geq 0$,

and so on, to give $A_n - G_n \geq 0$; ie $A_n \geq G_n$ for all n, as required.

[The Official mark scheme says:

“If $A_k = G_k$ ... then $A_{k-1} \geq G_{k-1}$, and by (i) and (ii) $A_{k-1} = G_{k-1}$”

It might be worth making the last deduction a bit clearer; eg:

“From (*), with $k - 1$ in place of k,

$k(A_k - G_k) \geq (k-1)(A_{k-1} - G_{k-1})$,

so that, as $A_k = G_k$, $A_{k-1} - G_{k-1} \leq 0$.

Thus $A_{k-1} \leq G_{k-1}$, and as $A_{k-1} \geq G_{k-1}$ it follows that

$A_{k-1} = G_{k-1}$.]

(b) [The question seems to be a bit ambiguous: is it essential to deduce (iii)(b) from the 2nd Part of (ii)? Or are we allowed to deduce (iii)(b) from (iii)(a) (assuming this is possible)?

It is highly likely that the 2nd Part of (ii) is there for a reason, but it doesn’t seem possible to form the chain of reasoning that was employed in (ii) (as there is no reference to the situation of equality in (i)).

The best assumption to make is probably that use of the 2nd Part of (ii) is required (and necessary), but – as we can’t see how to use the 2nd Part of (ii) – we should start working with what we have; ie $A_n = G_n$ and any results already established; and expect to use the 2nd Part of (ii) somewhere along the way (as in fact happens).]

From (*) in (iii)(a), $(k+1)(A_{k+1}-G_{k+1}) \geq k(A_k - G_k)$

With $n = k+1 > 1$, and $A_n = G_n$, it follows that

$A_{n-1} - G_{n-1} \leq 0$

Then, as $A_{n-1} \geq G_{n-1}$ (from (iii)(a)), it follows that $A_{n-1} = G_{n-1}$.

Repeating the argument gives $A_{n-2} = G_{n-2}, \dots, A_1 = G_1$

Suppose now that $a_i = a_1$ for $i \leq r$,

$A_{r+1} = G_{r+1} \Rightarrow \frac{a_1+a_2+\cdots+a_{r+1}}{r+1} = (a_1 a_2 \dots a_{r+1})^{\frac{1}{r+1}}$

$\left(\frac{ra_1+a_{r+1}}{r+1}\right)^{r+1} = {a_1}^r a_{r+1}$

Writing $a_{r+1} = \alpha a_1$, ${a_1}^{r+1}(\frac{r+\alpha}{r+1})^{r+1} = {a_1}^{r+1}\alpha$,

so that $(\frac{r+\alpha}{r+1})^{r+1} = \alpha$ (**)

Writing $x = \frac{r+\alpha}{r+1}$ and $k = r$,

(**) becomes $f(x) = x^{k+1} - [(k+1)x - k] = 0$

or $f(x) = x^{k+1} - (k+1)x + k = 0$,

and from the 2nd Part of (ii), $f(x) = 0$ if and only if $x = 1$;

ie when $\frac{r+\alpha}{r+1} = 1$, so that $\alpha = 1$

Thus, if $a_i = a_1$ for $i \leq r$, then $a_{r+1} = a_1$ also, provided that

$r + 1 \leq n$

Applying this in turn for $r = 2$ to $r = n$, gives the required result that $a_1 = a_2 = \cdots = a_n$

[In the Official mark scheme, after establishing that

$A_k = G_k$ and $a_k = G_{k-1}$ it says:

"But $A_1 = G_1 = a_1$ and so $a_2 = G_1 = a_1$ and thus $A_2 = G_2 = a_1$"

But it isn't clear where $G_2 = a_1$ comes from.]