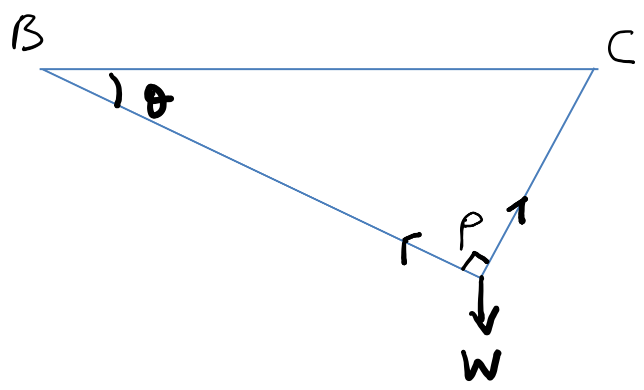
STEP 2022, P3, Q10 - Solution (4 pages; 12/6/24)

(i)



The tension in PB is $\frac{s_1W(PB-a)}{a}$ and similarly for PC.

Resolving the forces on P horizontally:

$\frac{s_1W(PB-a)}{a}.cos\theta = \frac{s_2W(PC-a)}{a}.sin\theta \quad (1)$

And vertically: $\frac{s_1W(PB-a)}{a}.sin\theta + \frac{s_2W(PC-a)}{a}.cos\theta = W \quad (2)$

Also $PB = 2acos\theta$ and $PC = 2asin\theta$

Substituting these into (1) & (2) gives:

$s_1(2cos\theta - 1).cos\theta = s_2(2sin\theta - 1)sin\theta \ \ (1')$

& $s_1(2cos\theta - 1).sin\theta + s_2(2sin\theta - 1)cos\theta = 1 \ \ (2')$

Writing $t_1 = s_1(2cos\theta - 1)$ & $t_2 = s_2(2sin\theta - 1)$,

$t_1cos\theta - t_2sin\theta = 0 \ \ (1'')$

& $t_1sin\theta + t_2cos\theta = 1 \ \ (2'')$

Then $\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \frac{1}{cos^2\theta + sin^2\theta}\begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} sin\theta \\ cos\theta \end{pmatrix}$

so that $s_1 = \frac{sin\theta}{2cos\theta - 1}$ and $s_2 = \frac{cos\theta}{2sin\theta - 1}$

(As $PB = 2acos\theta > a$ & $PC = 2asin\theta > a, 2cos\theta - 1 > 0$ &

$2sin\theta - 1 > 0$)

(ii) **1st Part**

The elastic potential energy in PB is $\frac{1}{2}(\frac{s_1 W}{a})(PB - a)^2$

$= \frac{sin\theta W(2acos\theta - a)^2}{2a(2cos\theta - 1)} = \frac{sin\theta Wa(2cos\theta - 1)}{2}$

and for PC it is $\frac{cos\theta W(2asin\theta - a)^2}{2a(2sin\theta - 1)} = \frac{cos\theta Wa(2sin\theta - 1)}{2}$

The GPE of P is $-W.PBsin\theta = -W.2acos\theta sin\theta$

So the total PE is

$-Wa(-\frac{sin\theta(2cos\theta - 1)}{2} - \frac{cos\theta(2sin\theta - 1)}{2} + 2cos\theta sin\theta)$

$= -Wa(-2sin\theta cos\theta + \frac{1}{2}sin\theta + \frac{1}{2}cos\theta + 2cos\theta sin\theta)$

$= -\frac{Wa(sin\theta + cos\theta)}{2}$ or $-paW$, where $p = \frac{sin\theta + cos\theta}{2}$

Now let $f(\theta) = sin\theta + cos\theta$

(noting that $f(0) = 1$ and $f\left(\frac{\pi}{2}\right) = 1$)

Then $f'(\theta) = cos\theta - sin\theta$

Now $cos\theta \neq 0$, as $0 < \theta < \frac{\pi}{2}$, so that $tan\theta$ is defined.

Then $f'(\theta) = 0 \Rightarrow cos\theta(1 - tan\theta) = 0$, so that $tan\theta = 1$

and hence $\theta = \frac{\pi}{4}$ (as $0 < \theta < \frac{\pi}{2}$)

and $f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$

Then, as $f''(\theta) = -sin\theta - cos\theta$, so that $f''\left(\frac{\pi}{4}\right) < 0$,

the maximum value for $p = \frac{f(\theta)}{2}$ is $\frac{\sqrt{2}}{2}$, provided that the

expressions for the elastic PE are both positive

(as there will be some extension of the two springs, assuming that

$W > 0$); which means that we require:

$sin\theta(2cos\theta - 1) > 0$ and $cos\theta(2sin\theta - 1) > 0$

When $\theta = \frac{\pi}{4}$, the LHS of these two inequalities are both

$\frac{1}{\sqrt{2}}\left(\frac{2}{\sqrt{2}} - 1\right) = \frac{1}{\sqrt{2}}(\sqrt{2} - 1) > 0$,

so that this requirement is met, and $p \leq \frac{\sqrt{2}}{2}$, as required.

The smallest value of p occurs when $f(\theta)$ is minimised, provided

that the value of θ is valid.

In the interval $0 \leq \theta \leq \frac{\pi}{2}$, this occurs when $\theta = 0$ or $\frac{\pi}{2}$, when

$p = \frac{f(\theta)}{2} = \frac{1}{2}$; so $p > \frac{1}{2}$, as $\theta = 0$ and $\frac{\pi}{2}$ are not attainable.

But, once again, the expressions for the elastic PE are required to

be positive, so that

$sin\theta(2cos\theta - 1) > 0$ and $cos\theta(2sin\theta - 1) > 0$

Hence $2cos\theta - 1 > 0$ and $2sin\theta - 1$ (as $sin\theta > 0$ *and* $cos\theta$ when

$0 < \theta < \frac{\pi}{2}$);

ie $cos\theta > \frac{1}{2}$ and $sin\theta > \frac{1}{2}$, whilst $f(\theta) = sin\theta + cos\theta$ is minimised;

so $\frac{\pi}{6} < \theta < \frac{\pi}{3}$, and therefore $f(\theta) > \sin\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{6}\right)$

(which equals $\sin\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{3}\right)$)

Hence $p = \frac{f(\theta)}{2} > \frac{\frac{1}{2}+\frac{\sqrt{3}}{2}}{2} = \frac{1}{4}(1+\sqrt{3})$, as required.

2nd Part

So $\frac{1}{2}\sqrt{2} \geq p > \frac{1}{4}(1+\sqrt{3})$

We want to show that $0.65 \leq p < 0.75$

[It isn't clear how much we are allowed to know about $\sqrt{3}$ (eg that $\sqrt{3} \approx 1.732$), and similarly for $\sqrt{2}$. The following approach assumes no knowledge at all.]

This will be achieved if we can show that $\frac{1}{2}\sqrt{2} < 0.75$ and that $\frac{1}{4}\left(1+\sqrt{3}\right) > 0.65$;

or that $\sqrt{2} < \frac{3}{2}$ and $1 + \sqrt{3} > 2.6$

As $y = x^2$ is an increasing function,

this is equivalent to showing that $2 < \frac{9}{4}$

and $3 > (1.6)^2 = 1 + 2(1)(0.6) + 0.36 = 2.56$,

both of which hold.

Thus $0.65 \leq p < 0.75$ and hence $p = 0.7$ to 1 sf.