STEP 2022, P3, Q7 - Solution (4 pages; 9/6/24)

(i) **1st Part**

$$f(r) = \begin{vmatrix} \underline{i} & a & x \\ \underline{j} & b & y \\ \underline{k} & c & z \end{vmatrix} = \begin{pmatrix} bz - cy \\ -(az - cx) \\ ay - bx \end{pmatrix}$$

Then $f(f(r)) = \begin{vmatrix} \underline{i} & a & bz - cy \\ \underline{j} & b & cx - az \\ \underline{k} & c & ay - bx \end{vmatrix}$

so that the x component of $f(f(r))$ is

$b(ay - bx) - c(cx - az)$

$= -x(b^2 + c^2) + aby + acz,$ as required

2nd Part

By symmetry, $f\big(f(r)\big) = \begin{pmatrix} -x(b^2 + c^2) + aby + acz \\ -y(a^2 + c^2) + bax + bcz \\ -z(a^2 + b^2) + cax + cby \end{pmatrix}$

As $\underline{n}$ is a unit vector, $a^2 + b^2 + c^2 = 1,$

and so $f\big(f(r)\big) = \begin{pmatrix} -x(a^2 + b^2 + c^2) + aby + acz + a^2x \\ -y(a^2 + b^2 + c^2) + bax + bcz + b^2y \\ -z(a^2 + b^2 + c^2) + cax + cby + c^2z \end{pmatrix}$

$$= \begin{pmatrix} -x + aby + acz + a^2x \\ -y + bax + bcz + b^2y \\ -z + cax + cby + c^2z \end{pmatrix}$$

$$= \begin{pmatrix} (ax + by + cz)a \\ (ax + by + cz)b \\ (ax + by + cz)c \end{pmatrix} - \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$= (\underline{n}\,.\,\underline{r})\underline{n} - \underline{r}$, as required.

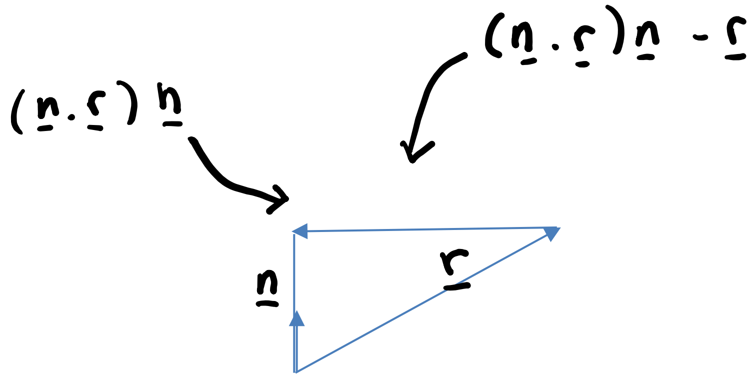
3rd Part

$(\underline{n}\,.\,\underline{r})\underline{n}$ is the vector component of $\underline{r}$ in the direction of $\underline{n}$

$(\underline{n}\,.\,\underline{r})\underline{n} - \underline{r}$ is perpendicular to $\underline{n}$,

as $[(\underline{n}\,.\,\underline{r})\underline{n} - \underline{r}]\,.\,\underline{n} = (\underline{n}\,.\,\underline{r})(1) - \underline{r}\,.\,\underline{n} = 0$

Thus the vectors $\underline{r}$, $(\underline{n}\,.\,\underline{r})\underline{n}$ and $(\underline{n}\,.\,\underline{r})\underline{n} - \underline{r}$ form a right-angled triangle.



(ii) [The wording of the question can be read in two ways:

(a) "By considering both $g(\underline{n})$ and $g(\underline{r})$, when $\underline{r}$ is perpendicular to $\underline{n}$ ..." , or

(b) "By considering $g(\underline{n})$, and then, when $\underline{r}$ is perpendicular to $\underline{n}$, considering $g(\underline{r})$, ..."

However, only (b) makes sense, as with $g(\underline{n})$, $\underline{r}$ is being set equal to $\underline{n}$, and so can't be perpendicular to $\underline{n}$.]

First of all, $g(\underline{n}) = \underline{n} + sin\theta\underline{n} \times \underline{n} + (1 - cos\theta)[(\underline{n} \,.\, \underline{n})\underline{n} - \underline{n}\,]$

$= \underline{n} + \underline{0} + \underline{0} = \underline{n}$ (*)

Then, when $\underline{r}$ is perpendicular to $\underline{n}$,

$g(\underline{r}) = \underline{r} + sin\theta\underline{n} \times \underline{r} + (1 - cos\theta)[(\underline{n} \,.\, \underline{r})\underline{n} - \underline{r}\,]$

$= \underline{r} + sin\theta\underline{t} - (1 - cos\theta)\underline{r}$,

where $\underline{t}$ is perpendicular to $\underline{n}$ *and* $\underline{r}$, and has the same magnitude as $\underline{r}$

$= sin\theta\underline{t} + cos\theta\underline{r}$ (**)

Consider a plane with normal $\underline{n}$. If $\underline{r}$ is perpendicular to $\underline{n}$, then $\underline{r}$ lies in this plane, and $\underline{t}$ also lies in the plane, and is perpendicular to $\underline{r}$, such that $\underline{n}$, $\underline{r}$ and $\underline{t} = \underline{n} \times \underline{r}$ form a right-handed set of mutually perpendicular axes (as for $\underline{i}$, $\underline{j}$ and $\underline{k}$), and therefore $\underline{r}$, $\underline{t}$ and $\underline{n}$ also form a right-handed set.

Now, any position vector $\underline{r}$ can be resolved into two components: one parallel to $\underline{n}$ and one perpendicular to $\underline{n}$ (ie in the plane with normal $\underline{n}$); thus $\underline{r} = a\underline{n} + \underline{r}'$

Then $g(\underline{r}) = (a\underline{n} + \underline{r}') + sin\theta\underline{n} \times (a\underline{n} + \underline{r}')$

$+(1 - cos\theta)[(\underline{n} \,.\, [a\underline{n} + \underline{r}'])\underline{n} - (a\underline{n} + \underline{r}')\,]$

$= ag(\underline{n}) + g(\underline{r}')$

$= a\underline{n} + (cos\theta\underline{r}' + sin\theta\underline{t}')$, from (*) and (**),

where $\underline{r}'$, $\underline{t}'$ and $\underline{n}$ form a right-handed set, and $|\underline{t}'| = |\underline{r}'|$

Now, as $\begin{pmatrix}1\\0\end{pmatrix}$ maps to $\begin{pmatrix}cos\theta\\sin\theta\end{pmatrix}$ under an anti-clockwise rotation of θ (about the Origin), $g(\underline{r})$ is the result of a rotation of $\underline{r}$ through an anti-clockwise angle θ , about an axis through the Origin in the direction of $\underline{n}$ (the component of $\underline{r}$ in the direction of $\underline{n}$ being left unchanged).

(iii) Consider $h(\underline{n}) = -\underline{n} - 2[(\underline{n}\,.\,\underline{n})\underline{n} - \underline{n}\,]$

$= -\underline{n} - 2(\underline{0}) = -\underline{n}$

And when $\underline{r}$ is perpendicular to $\underline{n}$,

$h(\underline{r}) = -\underline{r} - 2[(\underline{n}\,.\,\underline{r})\underline{n} - \underline{r}\,] = -\underline{r} - 2(-\underline{r}) = \underline{r}$

Once again, any position vector $\underline{r}$ can be resolved into components parallel and perpendicular to $\underline{n}$.

So Q is the reflection of R in the plane through the Origin with normal $\underline{n}$.

Alternatively:

For general $\underline{r}$,

$h(\underline{r}) = -\underline{r} - 2[(\underline{n}\,.\,\underline{r})\underline{n} - \underline{r}\,] = \underline{r} - 2(\underline{n}\,.\,\underline{r})\underline{n}$

As $(\underline{n}\,.\,\underline{r})\underline{n}$ is the projection of $\underline{r}$ onto $\underline{n}$, the effect of subtracting $2(\underline{n}\,.\,\underline{r})\underline{n}$ from $\underline{r}$ is to reflect R in the plane through the Origin with normal $\underline{n}$.