STEP 2022, P3, Q9 - Solution (4 pages; 4/6/24)

(i) **1st Part**

By Conservation of momentum, $mu = mv_1 + kmv_2$,

so that $u = v_1 + kv_2$ (1)

By NLR, $\frac{v_2 - v_1}{u - 0} = e$, so that $v_2 - v_1 = eu$ (2)

Then, substituting for v_1 from (1), $v_2 - (u - kv_2\ ) = eu$,

so that $v_2(1 + k) = u(e + 1)$, and hence $v_2 = u.\frac{e+1}{1+k}$

or $v_2 = \beta u$, where $\beta = \frac{1+e}{k+1}$

And then, from (2), $v_1 = v_2 - eu = (\beta - e)u = \alpha u$,

where $\alpha = \frac{1+e}{k+1} - e = \frac{1+e-e(k+1)}{k+1} = \frac{1-ke}{k+1}$

2nd Part

Let T_1 be the time between the 1st collision between A & B, and B hitting the wall; and let T_2 be the time between B hitting the wall and the 2nd collision between A & B.

Then $v_1(T_1 + T_2) = \frac{D}{2}$, $v_2T_1 = D$ and $ev_2T_2 = \frac{D}{2}$

Substituting for T_1 & T_2 from the 2nd & 3rd of the above eq'ns into the 1st:

$v_1(\frac{D}{v_2}+\frac{D}{2ev_2})=\frac{D}{2}$, so that $v_1.\frac{2e+1}{2ev_2}=\frac{1}{2}$ or $(2e+1)v_1=ev_2$

Then, from the 1st Part, $(2e+1)\alpha u=e\beta u$,

so that $(2e+1).\frac{1-ke}{k+1}=e.\frac{1+e}{k+1}$

or $(2e+1)(1-ke)=e(1+e)$

Then $1-ke=\frac{e(1+e)}{2e+1}$,

so that $ke=1-\frac{e(1+e)}{2e+1}=\frac{2e+1-e(1+e)}{2e+1}$

and hence $k=\frac{1+e-e^2}{e(2e+1)}$, as required.

(ii) The 2nd Part of (i) can be applied to the collision between B and C, as the ratio of the masses is k (A & B becoming B & C, and D replaced with $3d$). Thus we require $k=\frac{1+e-e^2}{e(2e+1)}$

Let T_1 be the time between the 1st collision between A & B, and B colliding with C; and let T_2 be the time between B colliding with C, and the 2nd collision between A & B.

Then $v_1(T_1+T_2)=d+\frac{3d}{2}$, $v_2T_1=d$ and $\alpha v_2T_2=\frac{3d}{2}$,

as, from the 1st Part of (i), the speed of the B after colliding with C will be reduced by the factor $\alpha=\frac{1-ke}{k+1}$ (B now taking the place of A).

Substituting for T_1 & T_2 from the 2nd & 3rd of the above eq'ns into the 1st:

$v_1(\frac{d}{v_2}+\frac{3d}{2\alpha v_2})=\frac{5d}{2}$, so that $v_1.\frac{2\alpha+3}{2\alpha v_2}=\frac{5}{2}$ or $(2\alpha+3)v_1=5\alpha v_2$

Also, from the 1st Part of (i), $v_1=\frac{1-ke}{k+1}u$ and $v_2=\frac{1+e}{k+1}u$

Hence $\frac{v_2}{v_1}=\frac{2\alpha+3}{5\alpha}$ and $\frac{v_2}{v_1}=\frac{1+e}{1-ke}$,

so that $\frac{2\alpha+3}{5\alpha}=\frac{1+e}{1-ke}$

Then, as $\alpha=\frac{1-ke}{k+1}$, it follows that

$\frac{2(1-ke)+3(k+1)}{5(1-ke)}=\frac{1+e}{1-ke}$

$\Rightarrow k(3-2e)+5=5(1+e)$

$\Rightarrow k=\frac{5e}{3-2e}$

Then, as we require $k=\frac{1+e-e^2}{e(2e+1)}$,

$\frac{5e}{3-2e}=\frac{1+e-e^2}{e(2e+1)}$

$\Rightarrow 5e^2(2e+1)=(3-2e)(1+e-e^2)$

[we can establish that $e=\frac{1}{2}$ satisfies this eq'n]

$\Rightarrow e^3(10-2)+e^2(5+3+2)+e(-3+2)-3=0$

ie $8e^3+10e^2-e-3=0$

or $(2e-1)(4e^2+7e+3)=0$

As $0<e<1,\ 4e^2+7e+3>0$,

and so the only sol'n is $e=\frac{1}{2}$, as was to be shown.