STEP 2023, P2, Q1 - Solution (5 pages; 17/6/24)

(i) Let $I = \int_a^b \frac{1}{(1+x^2)^{\frac{3}{2}}}\ dx$

With $x = \frac{1}{t}$, $dx = -\frac{1}{t^2}dt$, and $I = \int_{a^{-1}}^{b^{-1}} \frac{-\frac{1}{t^2}}{(1+\frac{1}{t^2})^{\frac{3}{2}}}\ dt$

(As $a > 0$ & $b > 0$, t is defined throughout.)

$= \int_{a^{-1}}^{b^{-1}} \frac{-t}{(t^2+1)^{\frac{3}{2}}}\ dt$, which gives the required integral.

(ii)(a) From (i), integral $= \int_2^{\frac{1}{2}} \frac{-t}{(t^2+1)^{\frac{3}{2}}}\ dt$

$= \int_{\frac{1}{2}}^{2} \frac{t}{(t^2+1)^{\frac{3}{2}}}\ dt$

Let $u = t^2$, so that $du = 2t\ dt$,

and the integral becomes $\int_{\frac{1}{4}}^{4} \frac{\frac{1}{2}}{(u+1)^{\frac{3}{2}}}\ du$

$= \frac{1}{2}\left[\frac{1}{(-\frac{1}{2})}(u+1)^{-\frac{1}{2}}\right]_{\frac{1}{4}}^{4}$

$= -\left(\frac{1}{\sqrt{5}} - \frac{1}{\sqrt{\frac{5}{4}}}\right) = \frac{1}{\sqrt{5}}$

(b) As $\frac{1}{(1+x^2)^{\frac{3}{2}}}$ is an even function , integral $= 2\int_0^2 \frac{1}{(1+x^2)^{\frac{3}{2}}}\ dx$

$= 2\lim_{c\to 0} \int_c^2 \frac{1}{(1+x^2)^{\frac{3}{2}}}\ dx$ (*)

With $x = \frac{1}{t}$, $dx = -\frac{1}{t^2}dt$,

$(*) = 2\lim_{d\to\infty}\int_d^{\frac{1}{2}}\frac{-\frac{1}{t^2}}{(1+\frac{1}{t^2})^{\frac{3}{2}}}\ dt$

$= 2\lim_{d\to\infty}\int_{\frac{1}{2}}^{\infty}\frac{t}{(t^2+1)^{\frac{3}{2}}}\ dt$ (**)

Let $u = t^2$, so that $du = 2t\ dt$,

and (**) becomes $2\lim_{d\to\infty}\int_{\frac{1}{4}}^{d^2}\frac{\frac{1}{2}}{(u+1)^{\frac{3}{2}}}\ du$

$= \lim_{d\to\infty}[\frac{1}{(-\frac{1}{2})}(u+1)^{-\frac{1}{2}}]_{\frac{1}{4}}^{d^2}$

$= -2\left(0 - \frac{1}{\sqrt{\frac{5}{4}}}\right) = \frac{4}{\sqrt{5}}$ or $\frac{4\sqrt{5}}{5}$

[$\int_{-2}^{0}\frac{1}{(1+x^2)^{\frac{3}{2}}}\ dx = \lim_{c\to 0^-}\int_{-2}^{c}\frac{1}{(1+x^2)^{\frac{3}{2}}}\ dx$ (#)

With $x = \frac{1}{t}$, $dx = -\frac{1}{t^2}dt$,

$(\#) = \lim_{d\to-\infty}\int_{-\frac{1}{2}}^{d}\frac{-\frac{1}{t^2}}{(1+\frac{1}{t^2})^{\frac{3}{2}}}\ dt$

$= \lim_{d\to-\infty}\int_{d}^{-\frac{1}{2}}\frac{t}{t^3(1+\frac{1}{t^2})^{\frac{3}{2}}}\ dt$ (##)

For the domain of the integral, $t < 0$ and $\frac{t}{t^3(1+\frac{1}{t^2})^{\frac{3}{2}}}$ cannot be written as $\frac{t}{(t^2+1)^{\frac{3}{2}}}$, as $t^3(1+\frac{1}{t^2})^{\frac{3}{2}} < 0$ whereas

$(t^2+1)^{\frac{3}{2}} > 0$ ($t^3 = t^{2\times\frac{3}{2}}$, but this doesn't equal $(t^2)^{\frac{3}{2}}$ if $t < 0$;

in general, $t^{ab} = (t^a)^b$ is not valid for $t < 0$ unless both a & b are integers)

Writing $T = -t$, so that $dT = -dt$,

(##) becomes $\lim\limits_{d\to-\infty} \int_{-d}^{\frac{1}{2}} \frac{-T}{-T^3(1+\frac{1}{T^2})^{\frac{3}{2}}}\ (-1)dT$

$= \lim\limits_{D\to\infty} \int_{\frac{1}{2}}^{D} \frac{T}{T^3(1+\frac{1}{T^2})^{\frac{3}{2}}}\ dT$

$= \lim\limits_{D\to\infty} \int_{\frac{1}{2}}^{D} \frac{T}{(T^2+1)^{\frac{3}{2}}}\ dT$ (###)

Let $u = T^2$, so that $du = 2T\ dT$,

and (###) becomes $\lim\limits_{D\to\infty} \int_{\frac{1}{4}}^{\infty} \frac{\frac{1}{2}}{(u+1)^{\frac{3}{2}}}\ du$

$= \frac{1}{2} \lim\limits_{D\to\infty} [\frac{1}{(-\frac{1}{2})}(u+1)^{-\frac{1}{2}}]_{\frac{1}{4}}^{\infty}$

$= -\lim\limits_{D\to\infty} (\frac{1}{\sqrt{D^2+1}} - \frac{1}{\sqrt{\frac{5}{4}}})$

$= \frac{1}{\sqrt{\frac{5}{4}}} - 0 = \frac{2}{\sqrt{5}}$]

(iii)(a) **1st Part**

Consider $J = \int_{\frac{1}{2}}^{2} \frac{x^2}{(1+x^2)^2}\ dx$

Let $t = \frac{1}{x}$, so that $dt = -\frac{1}{x^2}\ dx$

Then $J = \int_2^{\frac{1}{2}} \frac{\left(\frac{1}{t^2}\right)}{\left(1+\left(\frac{1}{t^2}\right)\right)^2} (-\frac{1}{t^2})\ dt$

$= \int_{\frac{1}{2}}^2 \frac{1}{(t^2+1)^2}\ dt = \int_{\frac{1}{2}}^2 \frac{1}{(1+x^2)^2}\ dx$

Thus, $\int_{\frac{1}{2}}^2 \frac{1}{(1+x^2)^2}\ dx = \int_{\frac{1}{2}}^2 \frac{x^2}{(1+x^2)^2}\ dx$, as required.

2nd Part

Now, $\int_{\frac{1}{2}}^2 \frac{x^2}{(1+x^2)^2}\ dx = \int_{\frac{1}{2}}^2 \frac{x^2+1}{(1+x^2)^2}\ dx - \int_{\frac{1}{2}}^2 \frac{1}{(1+x^2)^2}\ dx$

$= \int_{\frac{1}{2}}^2 \frac{1}{1+x^2}\ dx - \int_{\frac{1}{2}}^2 \frac{1}{(1+x^2)^2}\ dx$

Hence, $\int_{\frac{1}{2}}^2 \frac{1}{1+x^2}\ dx = \int_{\frac{1}{2}}^2 \frac{x^2}{(1+x^2)^2}\ dx + \int_{\frac{1}{2}}^2 \frac{1}{(1+x^2)^2}\ dx$

$= 2\int_{\frac{1}{2}}^2 \frac{x^2}{(1+x^2)^2}\ dx$, from the 1st Part ;

so that $\int_{\frac{1}{2}}^2 \frac{x^2}{(1+x^2)^2}\ dx = \frac{1}{2}\int_{\frac{1}{2}}^2 \frac{1}{1+x^2}\ dx$, as required.

3rd Part

Hence $\int_{\frac{1}{2}}^2 \frac{1}{(1+x^2)^2}\ dx = \frac{1}{2}\int_{\frac{1}{2}}^2 \frac{1}{1+x^2}\ dx$

$= \frac{1}{2}[arctanx]_{\frac{1}{2}}^{2}$

$= \frac{1}{2}(arctan2 - \left(\frac{\pi}{2} - arctan2\right))$

$= arctan2 - \frac{\pi}{4}$

(b) Write $I = \int_{\frac{1}{2}}^{2} \frac{1-x}{x(1+x^2)^{\frac{1}{2}}}\, dx$

Let $t = \frac{1}{x}$, so that $dt = -\frac{1}{x^2}\, dx$

Then $I = \int_{2}^{\frac{1}{2}} \frac{1-\frac{1}{t}}{\frac{1}{t}\left(1+(\frac{1}{t})^2\right)^{\frac{1}{2}}} \left(-\frac{1}{t^2}\right) dt$

$= \int_{\frac{1}{2}}^{2} \frac{t-1}{t(t^2+1)^{\frac{1}{2}}}\, dt = -I$, so that $I = 0$