STEP 2023, P2, Q12 - Solution (4 pages; 22/7/24)

(i) **1st Part**

If each $X_i \leq t$, then $Y \leq t$,

and if $Y \leq t$, it follows that each $X_i \leq t$ (otherwise $Y > t$).

So $Y \leq t \Leftrightarrow$ each $X_i \leq t$,

and hence $P(Y \leq t) = P(\text{each } X_i \leq t)$

$= P(X_1 \leq t)P(X_2 \leq t) \ldots P(X_n \leq t)$,

as the X_i are independent,

$= [P(X_1 \leq t)]^n$, as the X_i are identically distributed.

2nd Part

If $f_Y(t)$ is the pdf of Y, and $F_Y(t)$ its CDF [cumulative distribution function], then $f_Y(t) = \frac{d}{dt}F_Y(t)$ or $\frac{d}{dt}P(Y \leq t)$

$= \frac{d}{dt}[P(X_1 \leq t)]^n$

And $P(X_1 \leq t) = \int_0^t \frac{1}{2} sinx\, dx = \frac{1}{2}[-cosx]_0^t$

$= \frac{1}{2}(-cost + 1)$

So $f_Y(t) = \frac{d}{dt}[\frac{1}{2}(1 - cost)]^n$

$= \frac{n}{2^n}(1 - cost)^{n-1} sint$

(ii) **1st Part**

$\int_0^{m(n)} \frac{n}{2^n}(1-cost)^{n-1} sint\ dt = \frac{1}{2}$

Let $u = cost$, so that $du = -sint\ dt$,

and $\frac{n}{2^n}\int_1^{\cos[m(n)]}(1-u)^{n-1}\,(-1)du = \frac{1}{2}$,

so that $[(1-u)^n]_1^{\cos[m(n)]} = 2^{n-1}$,

and hence $(1-\cos[m(n)])^n = 2^{n-1}$,

so that $1-\cos[m(n)] = 2^{\frac{n-1}{n}}$,

and $\cos[m(n)] = 1-2^{1-\frac{1}{n}}$,

so that $m(n) = \arccos\,(1-2^{1-\frac{1}{n}})$, as $0 \le m(n) \le \pi$

2nd Part

$m(1) = \arccos(0) = \frac{\pi}{2}$

and as $n \to \infty$, $m(n) \to \arccos(-1) = \pi$

(as the maximum of the X_i will be distributed closer to π

as $n \to \infty$)

Thus, as n increases, $m(n)$ increases (with an upper limit of π).

(iii) **1st Part**

$\mu(n) = \int_0^{\pi} t(\frac{1}{2})^n n(1-cost)^{n-1} sint\ dt$

$= (\frac{1}{2})^n[t(1-cost)^n]_0^{\pi} - (\frac{1}{2})^n\int_0^{\pi}(1-cost)^n\ dt$, by Parts

$= (\frac{1}{2})^n[t(1-cost)^n]_0^{\pi} - (\frac{1}{2})^n \int_0^{\pi}(1-cost)^n\,dt$

$= \pi - \frac{1}{2^n}\int_0^{\pi}(1-cosx)^n\,dx$, as required (as $(1-cos\pi)^n = 2^n$)

(a) Result to prove: $\mu(n+1) > \mu(n)$;

ie that $\pi - \frac{1}{2^{n+1}}\int_0^{\pi}(1-cosx)^{n+1}\,dx > \pi - \frac{1}{2^n}\int_0^{\pi}(1-cosx)^n\,dx$,

or $\frac{1}{2^n}\int_0^{\pi}(1-cosx)^n\,dx - \frac{1}{2^{n+1}}\int_0^{\pi}(1-cosx)^{n+1}\,dx > 0$;

ie $\frac{1}{2^{n+1}}\int_0^{\pi}(1-cosx)^n(2-[1-cosx])\,dx > 0$

or $\int_0^{\pi}(1-cosx)^n(1+cosx)\,dx > 0$,

which holds, as the integrand is non-negative over the domain

$0 \le x \le \pi$.

(b) Result to prove: $\mu(2) < m(2)$;

ie $\pi - \frac{1}{2^2}\int_0^{\pi}(1-cosx)^2\,dx < \arccos\left(1-2^{1-\frac{1}{2}}\right)$;

or $\pi - \frac{1}{4}\int_0^{\pi}1-2cosx+cos^2x\,dx < \arccos\left(1-\sqrt{2}\right)$;

or $\pi - \frac{1}{4}\int_0^{\pi}1-2cosx+\frac{1}{2}(1+cos2x)dx < \pi - \arccos\left(\sqrt{2}-1\right)$;

or $\frac{1}{4}\left[\frac{3}{2}x - 2sinx + \frac{1}{4}sin2x\right]_0^{\pi} > \arccos\left(\sqrt{2}-1\right)$;

or $\arccos\left(\sqrt{2}-1\right) < \frac{3\pi}{8}$

or $\sqrt{2}-1 > \cos\left(\frac{3\pi}{8}\right)$, as the cosine function is decreasing for

angles between 0 and π;

or $\left(\sqrt{2}-1\right)^2 > cos^2\left(\frac{3\pi}{8}\right) = \frac{1}{2}\left(1+\cos\left(\frac{3\pi}{4}\right)\right)$;

or $2+1-2\sqrt{2} > \frac{1}{2}(1-\frac{\sqrt{2}}{2})$;

or $12-8\sqrt{2} > 2-\sqrt{2}$;

or $10 > 7\sqrt{2}$;

or $10^2 > 49(2)$;

ie $100 > 98$

Thus $\mu(2) < m(2)$