STEP 2023, P2, Q7 - Solution (4 pages; 23/6/24)

(i) $|z||w| = \sqrt{a^2+b^2}\sqrt{c^2+d^2} = \sqrt{(a^2+b^2)(c^2+d^2)}$

And $|zw| = |(a+bi)(c+di)| = |ac-bd+i(bc+ad)|$

$= \sqrt{(ac-bd)^2+(bc+ad)^2}$

$= \sqrt{a^2c^2+b^2d^2-2abcd+b^2c^2+a^2d^2+2abcd}$

$= \sqrt{a^2(c^2+d^2)+b^2(c^2+d^2)}$

$= \sqrt{(a^2+b^2)(c^2+d^2)}$

Thus $|zw| = |z||w|$.

(ii) From (i), $|(2+i)(10+11i)| = |2+i|\,.\,|10+11i|$

Now, $(2+i)(10+11i) = 20-11+i(22+10) = 9+32i$,

so that $|9+32i|^2 = |2+i|^2|10+11i|^2$;

ie $9^2+32^2 = 5\times 221$

Thus, $h = 9$ & $k = 32$ (or the other way round).

(iii) Now $8045 = 5\times 1609 = 5\times(40^2+3^2)$

And, from (i), $|(2+i)(40+3i)|^2 = |2+i|^2|40+3i|^2$,

so that $|80-3+i(40+6)|^2 = 5\times(40^2+3^2)$,

and hence $77^2+46^2 = 8045$

Thus, $m = 77$ & $n = 46$ (or the other way round).

(iv) From (i), $|(6+0i)(102+201i)|^2 = |6+0i|^2|102+201i|^2$,

so that $|612+1206i|^2 = 36 \times (102^2+201^2)$,

and hence $612^2 + 1206^2 = 36 \times 50805$

Thus, $p = 612$ & $q = 1206$ (or the other way round).

(v) Consider $(1000+a)^2 + 9^2$

Then $a = 1$, and $1001^2 + 9^2 = 1002082$

From (i), $|(5+0i)(1001+9i)|^2 = |5+0i|^2|1001+9i|^2$,

so that $|5005+45i|^2 = 25 \times (1001^2+9^2)$,

and hence $5005^2 + 45^2 = 25 \times 1002082$

Thus, $r = 45$ & $s = 5005$ is one solution.

[Note that $|(0+5i)(1001+9i)|^2 = |0+5i|^2|1001+9i|^2$

produces $|-45+5005i|^2 = 25 \times (1001^2+9^2)$,

which gives the same solution.]

Also, $|(3+4i)(1001+9i)|^2 = |3+4i|^2|1001+9i|^2$,

so that $|3003-36+i(4004+27)|^2 = 25 \times (1001^2+9^2)$,

and hence $2967^2 + 4031^2 = 25 \times 1002082$

Thus, $r = 2967$ & $s = 4031$ is another solution.

Finally, $|(4+3i)(1001+9i)|^2 = |4+3i|^2|1001+9i|^2$,

so that $|4004-27+i(3003+36)|^2 = 25 \times (1001^2+9^2)$,

and hence $3977^2 + 3039^2 = 25 \times 1002082$

Thus, $r = 3039$ & $s = 3977$ is a third solution.

(vi) We are given that

$109 \times 9193 = 1002037$ and $t^2 + u^2 = 9193$

Now, $109 = 10^2 + 3^2$ and $1002037 = 1001^2 + 6^2$

So we can write $(10^2 + 3^2) \times (t^2 + u^2) = 1001^2 + 6^2$,

From (i), $|(10 + 3i)(t + ui)|^2 = |10 + 3i|^2 |t + ui|^2$, for example.

This gives $|10t - 3u + i(3t + 10u)|^2 = (10^2 + 3^2) \times (t^2 + u^2)$

and hence $(10t - 3u)^2 + (3t + 10u)^2 = 1001^2 + 6^2$

Now $3t + 10u = 6$ has no positive integer solutions.

Suppose that $10t - 3u = 6$ and $3t + 10u = 1001$

ie $\begin{pmatrix} 10 & -3 \\ 3 & 10 \end{pmatrix} \begin{pmatrix} t \\ u \end{pmatrix} = \begin{pmatrix} 6 \\ 1001 \end{pmatrix}$,

so that $\begin{pmatrix} t \\ u \end{pmatrix} = \frac{1}{109} \begin{pmatrix} 10 & 3 \\ -3 & 10 \end{pmatrix} \begin{pmatrix} 6 \\ 1001 \end{pmatrix} = \frac{1}{109} \begin{pmatrix} 3063 \\ 9992 \end{pmatrix}$

but these are not integer solutions for t & u.

More generally, we can have the following:

$|(10 \pm 3i)(t \pm ui)|^2 = 1001^2 + 6^2$

[Other cases such as $|(-10 + 3i)(-t + ui)|^2$ can be rearranged into the above general form.]

So the resulting equations will be in one of the following forms:

$10t \pm 3u = 1001 \; or \; 6$ and $3t \pm 10u = 6 \; or \; 1001$

(where eg $+3u$ together with $-10u$ may be possible)

In matrix form,

$\begin{pmatrix} 10 & \pm 3 \\ 3 & \pm 10 \end{pmatrix} \begin{pmatrix} t \\ u \end{pmatrix} = \begin{pmatrix} 1001 \; or \; 6 \\ 6 \; or \; 1001 \end{pmatrix}$

so that $\binom{t}{u} = \frac{1}{(109\ or\ 91)}\begin{pmatrix} \pm 10 & \pm 3 \\ \pm 3 & \pm 10 \end{pmatrix}\begin{pmatrix} 1001\ or\ 6 \\ 6\ or\ 1001 \end{pmatrix}$

Now, $10 \times 1001 + 3 \times 6 = 10028$ and $10028 \div 109 = 92$

Also, $3 \times 1001 - 10 \times 6 = 2943$ and $2943 \div 109 = 27$

So we want $\binom{t}{u} = \frac{1}{109}\begin{pmatrix} 10 & 3 \\ 3 & -10 \end{pmatrix}\begin{pmatrix} 1001 \\ 6 \end{pmatrix} = \begin{pmatrix} 92 \\ 27 \end{pmatrix}$

(which originates from $|(-10 + 3i)(t - ui)|^2 = 1001^2 + 6^2$)

Thus, $t = 92$ & $u = 27$ (or the other way round).