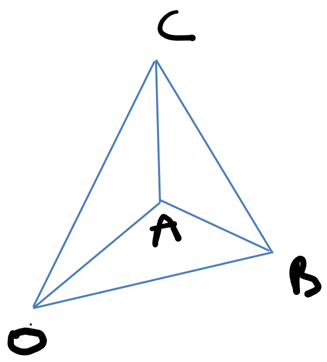
STEP 2023, P2, Q8 - Solution (5 pages; 20/7/24)

(i)



Referring to the diagram, the 6 edges are OA, OB, OC, AB, AC & BC

The pairs of edges that don't share a vertex are:

OA & BC, OB & AC, OC & AB

The 4 faces are: OAB, OAC, OBC & ABC,

and their perimeters are $OA + AB + BO$,

$OA + AC + CO,\ \ OB + BC + CO\ \ \&\ \ AB + BC + CA$

(where eg OA now denotes the length of edge OA, so that $AO = OA$)

Suppose that $OA = BC, OB = AC\ \&\ OC = AB$

Then $OA + AB + BO = OA + OC + OB$,

$OA + AC + CO = OA + OB + OC$,

$OB + BC + CO = OB + OA + OC$

and $AB + BC + CA = OC + OA + OB$

So, if a tetrahedron is isosceles, then its faces all have the same perimeter.

Suppose now that the faces all have the same perimeter, so that:

$OA + AB + BO = OA + AC + CO = OB + BC + CO$

$= AB + BC + CA$ (*)

(Result to prove: $OA = BC, OB = AC$ & $OC = AB$)

The eq'ns (*) simplify to:

$AB + BO = AC + CO$ (1)

$OA + AC = OB + BC$ (2)

and $OB + CO = AB + CA$ (3)

$OA - BC = OB - AC$ (from (2))

$= AB - CO$ (from (3))

Write $X = OA - BC = OB - AC = AB - CO$

So result to prove is $X = 0$.

Then, from (1), $AB - CO = AC - BO$; thus $X = -X$

Hence $X = 0$, as required.

(ii) **1st Part**

As the tetrahedron is isosceles, $OA = BC$;

ie $|\underline{a}| = |\underline{c} - \underline{b}|$, so that $|\underline{a}|^2 = |\underline{c} - \underline{b}|^2$,

and hence $|\underline{a}|^2 = (\underline{c} - \underline{b}).(\underline{c} - \underline{b}) = \underline{c}.\underline{c} - 2\underline{b}.\underline{c} + \underline{b}.\underline{b}$

$= |\underline{c}|^2 - 2\underline{b}.\underline{c} + |b|^2$,

so that $2\underline{b}.\underline{c} = |b|^2 + |c|^2 - |\underline{a}|^2$, as required.

2nd Part

By symmetry, $2\underline{a}.\underline{b} = |a|^2 + |b|^2 - |\underline{c}|^2$

and $2\underline{a}.\underline{c} = |a|^2 + |c|^2 - |\underline{b}|^2$

Then $2\underline{a}.\underline{b} + 2\underline{a}.\underline{c} = \left(|a|^2 + |b|^2 - |\underline{c}|^2\right) + (|a|^2 + |c|^2 - |\underline{b}|^2)$,

so that $2\underline{a}.(\underline{b} + \underline{c}) = 2|a|^2$,

and hence $\underline{a}.(\underline{b} + \underline{c}) = |a|^2$, as required.

(iii) The square of the distance of G from O is:

$\frac{1}{16}|\underline{a} + \underline{b} + \underline{c}|^2 = \frac{1}{16}(\underline{a} + \underline{b} + \underline{c}).(\underline{a} + \underline{b} + \underline{c})$

$= \frac{1}{16}([\underline{a}.\underline{a} + \underline{a}.(\underline{b} + \underline{c})] + [\underline{b}.\underline{b} + \underline{b}.(\underline{a} + \underline{c})] + [\underline{c}.\underline{c} + \underline{c}.(\underline{a} + \underline{b})])$

$= \frac{1}{16}([|a|^2 + |a|^2] + [|b|^2 + |b|^2] + [|c|^2 + |c|^2])$,

by the 2nd Part of (ii) and symmetry

$= \frac{1}{8}(|a|^2 + |b|^2 + |c|^2)$

The square of the distance of G from A is:

$\left|\frac{1}{4}(\underline{a}+\underline{b}+\underline{c})-\underline{a}\right|^2=\left|\frac{1}{4}(\underline{b}+\underline{c}-\underline{3a})\right|^2$

$=\frac{1}{16}(\underline{b}+\underline{c}-3\underline{a}).(\underline{b}+\underline{c}-3\underline{a})$

$=\frac{1}{16}([\underline{b}.\underline{b}+\underline{b}.\underline{c}-3\underline{b}.\underline{a}]+[\underline{c}.\underline{b}+\underline{c}.\underline{c}-3\underline{c}.\underline{a}]$

$+[-3\underline{a}.\underline{b}-3\underline{a}.\underline{c}+9\underline{a}.\underline{a}])$

$=\frac{1}{16}(9|a|^2+|b|^2+|c|^2-6\underline{a}.\underline{b}-6\underline{a}.\underline{c}+2\underline{b}.\underline{c})$

which, from the 1st Part of (ii) and symmetry

$=\frac{1}{16}(9|a|^2+|b|^2+|c|^2-3[|a|^2+|b|^2-|\underline{c}|^2]-3[|a|^2+|c|^2-$
$|\underline{b}|^2]+[|b|^2+|c|^2-|\underline{a}|^2])$

$=\frac{1}{16}(2|a|^2+2|b|^2+2|c|^2)$,

which thus equals the square of the distance of G from O,

and by symmetry the squares of the distance of G from B and C

have the same value; ie G is equidistant from all 4 vertices.

(iv) **1st Part**

Consider $|\underline{a}-\underline{b}-\underline{c}|^2$ [which will be non-negative]

$=(\underline{a}-\underline{b}-\underline{c}).(\underline{a}-\underline{b}-\underline{c})$

$=\underline{a}.\underline{a}-\underline{a}.(\underline{b}+\underline{c})-\underline{b}.\underline{a}+\underline{b}.\underline{b}+\underline{b}.\underline{c}-\underline{c}.\underline{a}+\underline{c}.\underline{b}+\underline{c}.\underline{c}$ (*)

From the 2nd Part of (ii), $\underline{a}.(\underline{b}+\underline{c})=|\underline{a}|^2$, so that

$\underline{a}.\underline{a}-\underline{a}.(\underline{b}+\underline{c})=|\underline{a}|^2-\underline{a}.(\underline{b}+\underline{c})=0$,

and (*) equals $-\underline{b}.\underline{a}+\underline{b}.\underline{b}+\underline{b}.\underline{c}-\underline{c}.\underline{a}+\underline{c}.\underline{b}+\underline{c}.\underline{c}$

$= -\underline{a}.(\underline{b} + \underline{c}) + |\underline{b}|^2 + 2\underline{b}.\underline{c} + |\underline{c}|^2$

$= -|\underline{a}|^2 + |\underline{b}|^2 + 2\underline{b}.\underline{c} + |\underline{c}|^2$, from the 2nd Part of (ii) again.

$= 4\underline{b}.\underline{c}$, from the 1st Part of (ii)

Then, as $|\underline{a} - \underline{b} - \underline{c}|^2 \geq 0$, $\underline{b}.\underline{c} \geq 0$, so that $\cos(BOC) = \frac{\underline{b}.\underline{c}}{|\underline{b}|.|\underline{c}|} \geq 0$,

and hence the angle between OB and OC cannot be obtuse.

And, by relabelling the vertices, the angle between any pair of edges that share a vertex cannot be obtuse.

2nd Part

Suppose that $\underline{b}.\underline{c} = 0$ (for example), so that the angle between OB and OC is a right angle.

Then, from the 1st Part of (iv), $|\underline{a} - \underline{b} - \underline{c}|^2 = 0$,

so that $|\underline{a} - \underline{b} - \underline{c}| = 0$, and hence $\underline{a} - \underline{b} - \underline{c} = \underline{0}$,

so that $\underline{a} = \underline{b} + \underline{c}$, which means that the vertices A, B & C lie in the same plane; which contradicts the fact that OABC is a tetrahedron.

Hence the angle between OB and OC cannot be a right angle; and, by relabelling the vertices, the angle between any pair of edges that share a vertex cannot be a right angle.