STEP 2023, P2, Q9 - Solution (5 pages; 25/7/24)

(i) **1st Part**

Applying N2L to the trailer, in the direction of motion, in the 3 situations:

$T_1 - \mu R_1 - kMgsin\alpha = kMa_1$, where $R_1 = kMgcos\alpha$ (1)

$T_2 - \mu R_2 = kMa_2$, where $R_2 = kMg$ (2)

$T_3 - \mu R_3 + kMgsin\alpha = kMa_3$, where $R_3 = kMgcos\alpha = R_1$ (3)

Writing $F = \mu Mg$, eq'ns (1)-(3) become:

$T_1 - kFcos\alpha - kMgsin\alpha = kMa_1$ $(1')$

$T_2 - kF = kMa_2$ $(2')$

$T_3 - kFcos\alpha + kMgsin\alpha = kMa_3$ $(3')$

[We could have written $F = k\mu Mg$, but it seems advantageous to keep the factor k in all terms, so that the T_i will be in terms of k.]

Applying N2L to the truck, in the direction of motion, in the 3 situations:

$D - T_1 - Mgsin\alpha = Ma_1$ (4)

$D - T_2 = Ma_2$ (5)

$D - T_3 + Mgsin\alpha = Ma_3$ (6)

[Avoiding using $(2')$ & (5), as these involve the unwanted T_2

(when trying to establish that $T_1 = T_3$). This also keeps it simple.]

Subtracting (6) from (4) [in order to eliminate D]:

$T_3 - T_1 - 2Mgsin\alpha = M(a_1 - a_3)$ (7)

Subtracting ($3'$) from ($1'$) [in order to eliminate $kFcos\alpha$]:

$T_1 - T_3 - 2kMgsin\alpha = kM(a_1 - a_3)$ (8)

Then, eliminating $2Mgsin\alpha$ from (7) & (8) gives:

$T_1 - T_3 + k(M(a_1 - a_3) + T_1 - T_3) = kM(a_1 - a_3)$,

and hence $(1 + k)(T_1 - T_3) = 0$,

so that $T_1 = T_3$, as required.

2nd Part

Adding eq'ns (4) & (6):

$2D - T_1 - T_3 = M(a_1 + a_3)$

Then, substituting for D from eq'n (5):

$2(Ma_2 + T_2) - T_1 - T_3 = M(a_1 + a_3)$,

so that $M(a_1 + a_3 - 2a_2) = 2(T_2 - T_1)$, as required (as $T_3 = T_1$).

(ii)(a) Subtracting eq'n (2) from eq'n (1):

$(T_1 - \mu kMgcos\alpha - kMgsin\alpha) - (T_2 - \mu kMg) = kMa_1 - kMa_2$

so that $T_1 - T_2 = kM(a_1 - a_2 + \mu g cos\alpha + g sin\alpha - \mu g)$

Then, from the 2nd Part of (i), $M(a_1 + a_3 - 2a_2) = 2(T_2 - T_1)$,

so that

$M(a_1 + a_3 - 2a_2) = 2kM(a_1 - a_2 + \mu g cos\alpha + g sin\alpha - \mu g)$

and so

$a_1 + a_3 - 2a_2 = 2k(a_1 - a_2 + \mu g cos\alpha + g sin\alpha - \mu g)$ (9)

Then, subtracting eq'n (2) from eq'n (3), and using the result that

$T_3 = T_1$:

$(T_1 - \mu kMg cos\alpha + kMg sin\alpha) - (T_2 - \mu kMg) = kMa_3 - kMa_2$,

so that $T_1 - T_2 = kM(a_3 - a_2 + \mu g cos\alpha - g sin\alpha - \mu g)$,

and so, using $M(a_1 + a_3 - 2a_2) = 2(T_2 - T_1)$ again,

$a_1 + a_3 - 2a_2 = 2k(a_3 - a_2 + \mu g cos\alpha - g sin\alpha - \mu g)$ (10)

Then, from (9) & (10), $a_1 - a_2 + \mu g cos\alpha + g sin\alpha - \mu g$

$= a_3 - a_2 + \mu g cos\alpha - g sin\alpha - \mu g$,

so that $a_1 - a_3 + 2g sin\alpha = 0$,

and hence $g = \frac{a_3 - a_1}{2 sin\alpha}$

This shows that $a_3 > a_1$, which is equivalent to $a_3 > \frac{1}{2}(a_1 + a_3)$

The lower inequality is equivalent to $a_1 + a_3 - 2a_2 > 0$,

which, from the 2nd Part of (ii), is equivalent to $T_2 > T_1$.

Thus, the result to prove is: $T_2 > T_1$

From (2), $T_2 - \mu kMg = kMa_2$

and from (5), $D - T_2 = Ma_2$

[This could enable us to obtain an expression for T_2, free of a_2. We might be able to use the fact that $D > 0$.]

Then $T_2 - \mu kMg = k(D - T_2)$,

so that $T_2(k+1) = kD + \mu kMg$

Also, from (1), $T_1 - \mu kMgcos\alpha - kMgsin\alpha = kMa_1$,

and from (4), $D - T_1 - Mgsin\alpha = Ma_1$

Then $T_1 - \mu kMgcos\alpha - kMgsin\alpha = k(D - T_1 - Mgsin\alpha)$

so that $T_1(k+1) = kD + \mu kMgcos\alpha$

Then $(T_2 - T_1)(k+1) = \mu kMg(1 - cos\alpha) > 0$ (as $0 < \alpha < \frac{\pi}{2}$),

and hence $T_2 > T_1$, as required.

[Note that we haven't used the fact that $\mu < 1$ yet!]

(b) Once again, from (2), $T_2 - \mu kMg = kMa_2$

and from (5), $D - T_2 = Ma_2$,

and adding these gives $D - \mu kMg = (k+1)Ma_2$

Also, again from (1), $T_1 - \mu kMgcos\alpha - kMgsin\alpha = kMa_1$,

and from (4), $D - T_1 - Mgsin\alpha = Ma_1$,

and adding these gives

$D - \mu kMgcos\alpha - kMgsin\alpha - Mgsin\alpha = (k+1)Ma_1$,

Then $(k+1)M(a_2 - a_1)$

$= -\mu kMg + \mu kMgcos\alpha + kMgsin\alpha + Mgsin\alpha$

$> kMg(-\mu + \mu cos\alpha + sin\alpha)$

$> kMg(-\mu + \mu cos\alpha + \mu sin\alpha)$, as $\mu < 1$

$= kMg\mu(cos\alpha + sin\alpha - 1)$

and so we will have shown that $a_1 < a_2$ if we can establish that

$cos\alpha + sin\alpha - 1 > 0$ for $0 < \alpha < \frac{\pi}{2}$

Now, $(cos\alpha + sin\alpha)^2 = cos^2\alpha + sin^2\alpha + 2cos\alpha sin\alpha$,

so that $(cos\alpha + sin\alpha)^2 = 1 + \sin(2\alpha) > 1$ for $0 < \alpha < \frac{\pi}{2}$,

and hence $cos\alpha + sin\alpha > 1$, as $cos\alpha + sin\alpha > 0$,

and so $cos\alpha + sin\alpha - 1 > 0$, as required,

and therefore $a_1 < a_2$.