STEP – Complex Numbers (14 pages; 12/4/24)

Quiz

(1) What is the imaginary part of $1 + 2i$?

(2) Classify the following as imaginary or complex:

(a) $1 + 2i$ (b) $2i$ (c) 0

(3) What is the geometric effect of:

(a) multiplication by i (b) multiplication by -1 (c) taking the conjugate

Answers

(1) 2

(2) (a) (non-real) complex (b) imaginary & complex

(c) imaginary & complex

(3) (a) rotation by $\frac{\pi}{2}$ anti-clockwise

(b) rotation by π (c) reflection in Real axis

Method for finding $\sqrt{24-10i}$?

Solution

To find $\sqrt{24-10i}$, let $24-10i=(a+bi)^2$; then equate real and imaginary parts.

Find $(1+i)^4$

Solution

$1 + i = \sqrt{2}e^{i\pi/4}$

Hence $(1 + i)^4 = 4e^{i\pi} = 4(cos\pi + isin\pi) = -4$

$p - q = ki(r + s) \Rightarrow p^* - q^* = ?$,

where k is real (and p, q, r & s are non-real complex numbers)

$[p - q = ki(r + s) \Rightarrow p^* - q^* = ?$,

where k is real (and p, q, r & s are non-real complex numbers)]

Solution

$([a + bi] + [c + di])^* = a + c - (b + d)i$

$= (a + bi)^* + (c + di)^*$

Also $[(a + bi)(c + di)]^* = [ac - bd + i(bc + ad)]^*$

$= ac - bd - i(bc + ad)$

and $(a + bi)^*(c + di)^* = (a - bi)(c - di)$

$= ac - bd - i(bc + ad)$

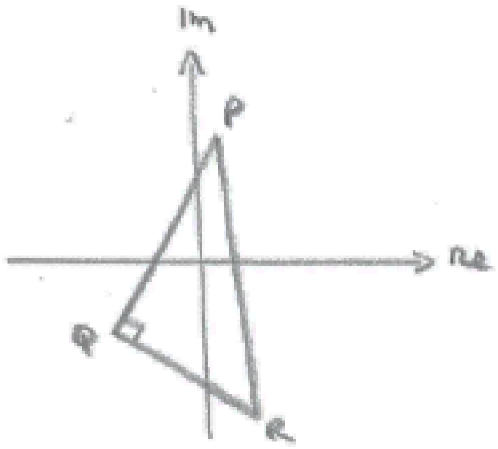
[Or consider conjugate as reflection in Real axis.]

$p^* - q^* = (p - q)^* = [ki(r + s)]^*$

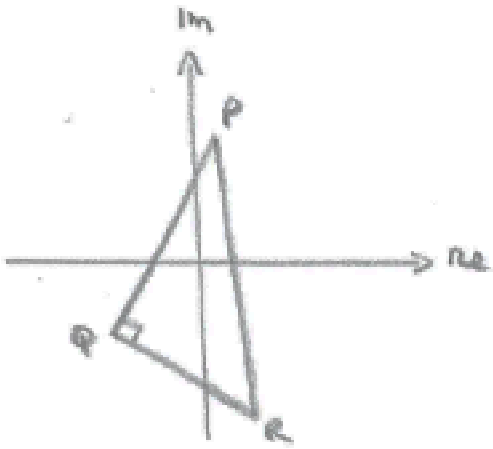
$= k^* i^* (r + s)^* = k(-i)(r^* + s^*)$

The corners of a right-angled triangle are the points P, Q & R in the Argand diagram (in anti-clockwise order, with the right-angle being at Q), represented by the complex numbers p, q & r.

Find an expression for $\frac{p-q}{r-q}$



Solution



$p - q = ki(r - q)$, where k is real;

so that $\frac{p-q}{r-q} = ki$

Find the modulus and argument of $e^{\frac{7\pi i}{10}} - e^{-\frac{9\pi i}{10}}$

[Find the modulus and argument of $e^{\frac{7\pi i}{10}} - e^{-\frac{9\pi i}{10}}$]

Solution

Write $z = e^{\frac{7\pi i}{10}} - e^{-\frac{9\pi i}{10}}$ in the form $e^{a\pi i}(e^{b\pi i} - e^{-b\pi i})$

So $a + b = \frac{7}{10}$ & $a - b = -\frac{9}{10}$

Then $a = -\frac{1}{10}$ & $b = \frac{8}{10}$

and $e^{\frac{7\pi i}{10}} - e^{-\frac{9\pi i}{10}} = e^{-\frac{\pi i}{10}}(e^{\frac{8\pi i}{10}} - e^{-\frac{8\pi i}{10}})$

$= e^{-\frac{\pi i}{10}}(2isin\left(\frac{4\pi}{5}\right))$

Then $|z| = \left|e^{-\frac{\pi i}{10}}\right| \left|2isin\left(\frac{4\pi}{5}\right)\right| = (1)(2sin\left(\frac{4\pi}{5}\right))$

$= 2\sin\left(\pi - \frac{4\pi}{5}\right) = 2\sin\left(\frac{\pi}{5}\right)$

and $\arg(z) = \arg\left(e^{-\frac{\pi i}{10}}\right) + \arg\left(2isin\left(\frac{4\pi}{5}\right)\right)$

$= -\frac{\pi}{10} + \frac{\pi}{2} = \frac{4\pi}{10} = \frac{2\pi}{5}$

How are the complex numbers $cos\theta + isin\theta$ and $sin\theta + icos\theta$ related?

Solution

$sin\theta + icos\theta = \cos\left(\frac{\pi}{2} - \theta\right) + isin(\frac{\pi}{2} - \theta)$

As both complex numbers have a modulus of 1, $sin\theta + icos\theta$ is the reflection of $cos\theta + isin\theta$ in the line Re = Im