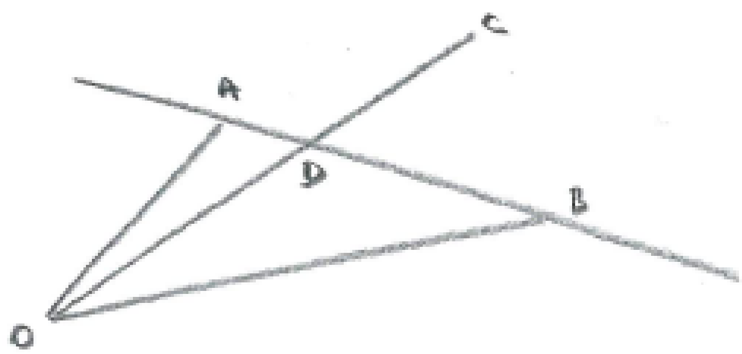
Vector Ideas (STEP) (8 pages; 9/6/24)

(1) Referring to the diagram below, if $\overrightarrow{OA} = \mathbf{a}$ etc, then

d can be written as λ**a** + (1-λ)**b**, where $\frac{DB}{AB} = \lambda$

(consider λ = 1, if in doubt)



(2) Again referring to the diagram in (1),

d can be written as μ**c**

In other words, the fact that D lies somewhere on the line OC can be used to generate an expression for **d**. This may not seem to be of much use, because an extra parameter has been introduced. However, the result in (3) enables two equations to be generated, so that the extra parameter can be eliminated, leaving one new equation!

Another example, also using the diagram in (1):

d can be written as **a** + μ$\overrightarrow{AB}$

This is obviously just a variant of (1), but is a quick way of taking account of the fact that D lies on the line through AB. See also the Example in (5) below.

This leads on to idea (3):

(3) Once again, referring to the diagram in (1):

One of the basic vector techniques is to find an alternative route to a particular point.

So, for example: $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD} = \overrightarrow{OA} + \lambda\overrightarrow{AB}$

This takes account of the fact that D lies on AB, without having to solve any simultaneous equations. See Example in (5) below.

(4) If **a** and **b** are non-parallel vectors, and p**a** + q**b** = r**a** + s**b**, then it follows that p = r and q = s.

We are used to using this for the unit vectors **i** and **j**, but it may appear to be counter-intuitive for other **a** and **b**: it seems as though we are equating coefficients of **a** and **b**, even though **a** and **b** are fixed (normally the method of equating coefficients is only valid if the relation is to hold for all values of the variable)! The reason it works for **i** and **j** is just that the vector equation

p**i** + q**j** = r**i** + s**j** is really a pair of equations: one for the **i** dimension and one for the **j** dimension, and the same applies to any **a** and **b** (provided they aren't parallel).

For example, referring again to the diagram in (1), if we are told that

c = α**a** + β**b**, then we have the following information about **d**:

(i) **d** = λ**a** + (1-λ)**b**

(ii) $\mathbf{d} = \mu(\alpha\mathbf{a} + \beta\mathbf{b})$

Then equating the coefficients of **a** and **b** gives:

$\lambda = \mu\alpha$ and $1 - \lambda = \mu\beta$

and μ can be eliminated to give the relation $\frac{\alpha}{\beta} = \frac{\lambda}{1-\lambda} = \frac{DB}{AD}$

See, for example, STEP Paper 2, 2007, Q8 (on which the above example is based).

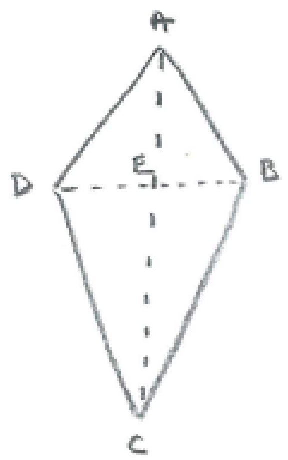
(5) Results involving lengths can often be obtained by use of the scalar product, as $PQ^2 = \left|\underline{q} - \underline{p}\right|^2 = \left(\underline{q} - \underline{p}\right).\left(\underline{q} - \underline{p}\right)$

[See STEP 2014, P3, Q7]

(6) Example [AEA, June 2009, Q7(d)]

In the diagram below, ABCD is a kite. Find $\overrightarrow{OD}$ if

$$\overrightarrow{OA} = \begin{pmatrix} -1 \\ 4/3 \\ 7 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 4 \\ 4/3 \\ 2 \end{pmatrix} \ \& \ \overrightarrow{OC} = \begin{pmatrix} 6 \\ 16/3 \\ 2 \end{pmatrix}$$



We need to take account of the special features of this case, namely that AC is perpendicular to BD (*) and bisects BD (**) (which enables D to be uniquely determined from A, B & C).

We can take an alternative route to D, in order to involve the other points, writing:

$\overrightarrow{OD} = \overrightarrow{OB} + 2\overrightarrow{BE}$ [this takes account of (**)]

To record the fact that E lies on AC, we write $\overrightarrow{BE} = \overrightarrow{BA} + \lambda\overrightarrow{AC}$

We also need to take account of (*): $\overrightarrow{BE}.\overrightarrow{AC} = 0$

Now $\overrightarrow{BA} = \begin{pmatrix} -5 \\ 0 \\ 5 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 7 \\ 4 \\ -5 \end{pmatrix}$

so that $\overrightarrow{BE} = \begin{pmatrix} -5 + 7\lambda \\ 4\lambda \\ 5 - 5\lambda \end{pmatrix}$

$\overrightarrow{BE}.\overrightarrow{AC} = 0$ gives $7(-5+7\lambda) + 4(4\lambda) - 5(5-5\lambda) = 0$

so that $90\lambda = 60$ and $\lambda = 2/3$

Hence

$\overrightarrow{BE} = \begin{pmatrix} -1/3 \\ 8/3 \\ 5/3 \end{pmatrix}$

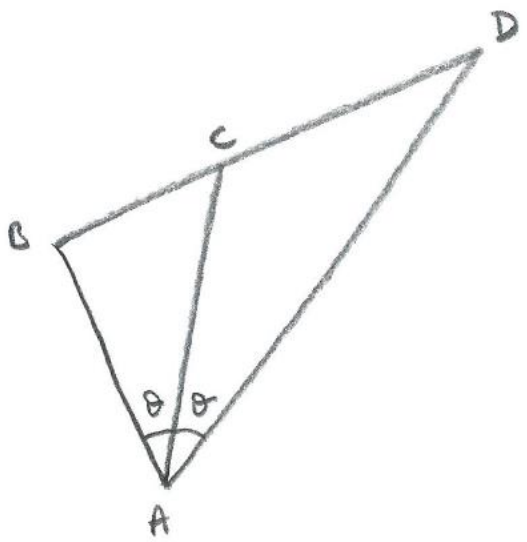
Then $\overrightarrow{OD} = \overrightarrow{OB} + 2\overrightarrow{BE} = \begin{pmatrix} 4 - 2/3 \\ \frac{4}{3} + 16/3 \\ 2 + 10/3 \end{pmatrix} = \begin{pmatrix} 10/3 \\ 20/3 \\ 16/3 \end{pmatrix} = 2/3 \begin{pmatrix} 5 \\ 10 \\ 18 \end{pmatrix}$

(7) The Sine and Cosine rules can sometimes be applied to triangles formed from vectors (ie the problem becomes one of simple geometry).

See, for example, STEP Paper 2, 2008, Q8 (1st part).

In particular, use can be made of the Angle Bisector Theorem

Referring to the diagram below, $\frac{CD}{AD} = \frac{BC}{AB}$

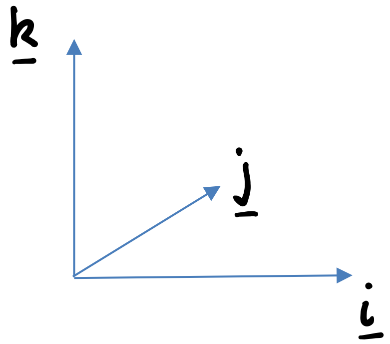


Proof

By the Sine rule, $\frac{BC}{sin\theta} = \frac{AB}{sinBCA}$, so that $\frac{BC}{AB} = \frac{sin\theta}{sinBCA}$

Also, $\frac{CD}{sin\theta} = \frac{AD}{sinACD} = \frac{AD}{sinBCA}$, and so $\frac{CD}{AD} = \frac{sin\theta}{sinBCA} = \frac{BC}{AB}$

(8) Direction of the vector product $\underline{a} \times \underline{b}$



To determine the direction of $\underline{a} \times \underline{b}$:

First of all, note that when undoing a nut threaded on a bolt, the nut has to be turned in an anti-clockwise direction (looking down the bolt towards the nut). The nut then moves up the thread towards you.

Consider the plane containing $\underline{a}$ and $\underline{b}$, and suppose that this is vertical and in front of you (eg the face of a laptop). Then if an anti-clockwise rotation is required to turn from $\underline{a}$ to $\underline{b}$ (eg $\underline{i} \times \underline{k}$), the direction of $\underline{a} \times \underline{b}$ will be towards you (so that $\underline{i} \times \underline{k}$ produces $-\underline{j}$). [Note that $\underline{a}$ and $\underline{b}$ don't need to be perpendicular to each other, but $\underline{a} \times \underline{b}$ will be perpendicular to the plane containing $\underline{a}$ and $\underline{b}$.]

If instead a clockwise rotation is required, then the direction of $\underline{a} \times \underline{b}$ will be away from you. For example, with $\underline{k} \times \underline{j}$ we can view the plane containing $\underline{k}$ and $\underline{j}$, with a clockwise rotation being required to turn from $\underline{k}$ to $\underline{j}$. Then the direction of $\underline{k} \times \underline{j}$ will be $-\underline{i}$.

(9) Vector Triple Product

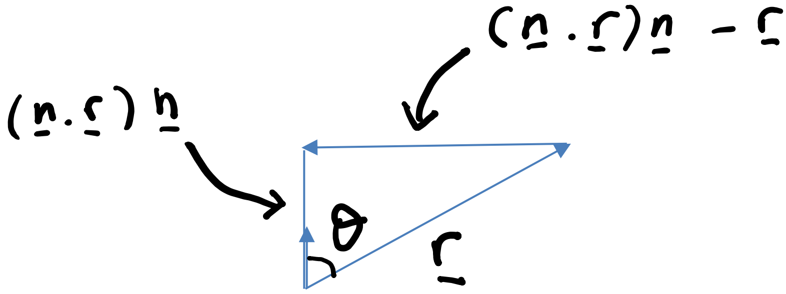
(i) It can be shown that $\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a}.\underline{c})\underline{b} - (\underline{a}.\underline{b})\underline{c}$

(eg by considering $\underline{b} \times \underline{c}$ as $\begin{vmatrix} \underline{i} & b_1 & c_1 \\ \underline{j} & b_2 & c_2 \\ \underline{k} & b_3 & c_3 \end{vmatrix}$)

(ii) Consider $\underline{n} \times (\underline{n} \times \underline{r})$, where $\underline{n}$ is the unit normal to a plane (which can be considered to be horizontal), and $\underline{r}$ is the position vector of a general point.

Then, from (i), $\underline{n} \times (\underline{n} \times \underline{r}) = (\underline{n}.\underline{r})\underline{n} - (\underline{n}.\underline{n})\underline{r}$

$= (\underline{n}.\underline{r})\underline{n} - \underline{r}$



$(\underline{n}.\underline{r})\underline{n}$ is the projection of $\underline{r}$ onto $\underline{n}$,

and $\underline{n} \times (\underline{n} \times \underline{r}) = (\underline{n}.\underline{r})\underline{n} - \underline{r}$ is directed towards the vertical axis $\underline{n}$, if $\underline{r}$ is considered to be the position vector of a point rotating about $\underline{n}$ in a horizontal plane. The magnitude of $\underline{n} \times (\underline{n} \times \underline{r})$ is $|\underline{r}|sin\theta$.

(iii) The direction of $\underline{n} \times (\underline{n} \times \underline{r})$ can also be deduced as follows:

$\underline{n} \times \underline{r}$ is in a horizontal plane, pointing into the paper (ie in the direction of $\underline{j}$). [See (8), noting that a clockwise rotation is needed to turn from $\underline{n}$ to $\underline{r}$.]

Then, as $\underline{n}$ and $\underline{n} \times \underline{r}$ are in the directions $\underline{k}$ and $\underline{j}$ respectively, $\underline{n} \times (\underline{n} \times \underline{r})$ will be in the direction of $-\underline{i}$. [See (8) again.]

(iv) Reflection of $\underline{r}$ in a horizontal plane with unit normal

$\underline{n}$ through the Origin: $\underline{r}$ transforms to $\underline{r} - 2(\underline{n}.\underline{r})\underline{n}$

$= \underline{r} - 2[\underline{n} \times (\underline{n} \times \underline{r}) + \underline{r}]$

$= -\underline{r} - 2\underline{n} \times (\underline{n} \times \underline{r})$