Simplex Algorithm - Part 2 (18 pages; 18/5/24)

Contents

(1) Constraints are of the ≥ type

The basic Simplex method (covered in Part 1) assumes that all the constraints are of the ≤ type (apart from $x \geq 0$, $y \geq 0$).

A constraint such as $3x + 2y - z \geq -2$ (as in Example 3 in Part 1) can just be rewritten as $-3x - 2y + z \leq 2$, but the constraint $3x + 2y - z \geq 2$ could not be dealt with in this way, as it would leave us with a negative value on the RHS.

There are two methods for dealing with such a problematic constraint: the 2-Stage Simplex method and the Big M (Simplex) method.

(2) 2-Stage Simplex Method

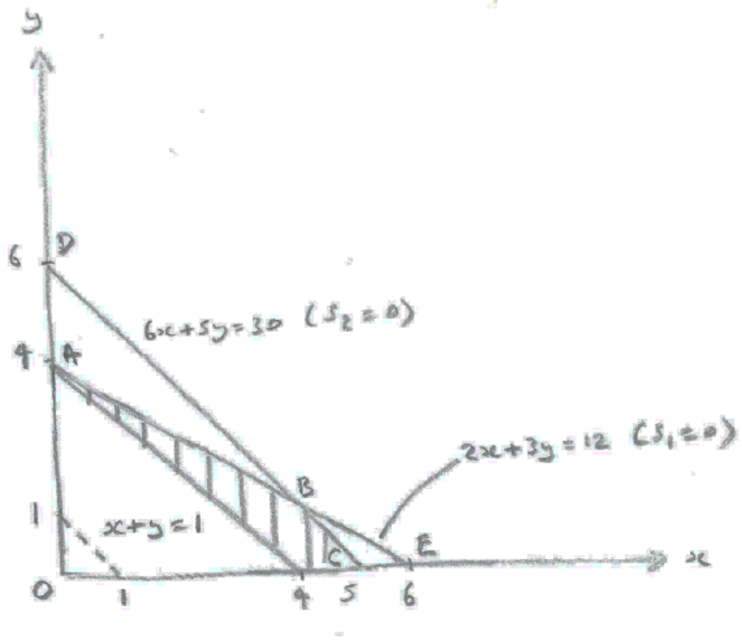
Example

Maximise $P = x + y$

subject to $2x + 3y \leq 12$

$6x + 5y \leq 30$

$x + y \geq 4$



Create slack and surplus variables as usual (for the $\leq$ *and* $\geq$ constraints respectively), but introduce an artificial variable for the $x + y \geq 4$ constraint.

$P - x - y \qquad\qquad = 0 \quad (1)$

$2x + 3y + s_1 \qquad = 12 \quad (2)$

$6x + 5y + \qquad s_2 = 30 \quad (3)$

$x + y - s_3 + a_1 = 4 \quad (4) \qquad\qquad (s_1,\ s_2, s_3,\ a_1 \geq 0)$

a_1 is needed because $x + y - s_3 = 4$ doesn't allow $x = y = 0$, since $s_3 \geq 0$; but with the artificial variable we can now start with $x = y = s_3 = 0$ & $a_1 = 4$

Initial solution:

$x = y = s_3 = 0$; $s_1 = 12,\ s_2 = 30,\ a_1 = 4,\ P = 0$

The aim is to minimise a_1 , so that (if possible) the solution moves into the feasible region.

Create a new objective: minimise $A = a_1$

[It might seem a bit unnecessary to create another variable with the same value as a_1, but it enables the method to be extended easily to cases where there are two or more artificial variables - see later example.]

From (4), re-write $A = a_1$ as $A + x + y - s_3 = 4$ (5)

Re-labelling the rows:

A	P	x	y	s_1	s_2	s_3	a_1		
1	0	1	1	0	0	-1	0	4	(1)
0	1	-1	-1	0	0	0	0	0	(2)
0	0	2	3	1	0	0	0	12	(3)
0	0	6	5	0	1	0	0	30	(4)
0	0	1	1	0	0	-1	1	4	(5)

Referring to the Simplex tableau above:

1st row: new objective (1st stage of method)

2nd row: original objective (2nd stage of method)

Each stage of the method involves applying the ordinary Simplex method.

To minimise A: the positive coefficients of x & y mean that x *or* y could be increased (alternatively maximise $-A$).

Choose x as the pivot column (eg) and apply the ratio test:

3rd row: $\frac{12}{2} = 6$, 4th row: $\frac{30}{6} = 5$, 5th row: $\frac{4}{1} = 4$

A	P	x	y	s_1	s_2	s_3	a_1		
1	0	0	0	0	0	0	-1	0	6 = 1 − 10
0	1	0	0	0	0	-1	1	4	7 = 2 + 10
0	0	0	1	1	0	2	-2	4	8 = 3 − 2 × 10
0	0	0	-1	0	1	6	-6	6	9 = 4 − 6 × 10
0	0	1	1	0	0	-1	1	4	10 = 5

As $A = 0$, the 1st stage has been successfully completed.

Now remove the 1st row, and the columns for A and a_1

(a_1 is a non-basic variable and is being set to 0)

Solution so far: $x = 4,\ y = 0,\ P = 4$

The 2nd stage is now to maximise P, as usual.

The remainder of the working is as follows:

P	x	y	s_1	s_2	s_3		
1	0	0	0	0	-1	4	7
0	0	1	1	0	2	4	8
0	0	-1	0	1	6	6	9
0	1	1	0	0	-1	4	10

Choose s_3 as the pivot column, and apply the ratio test:

8th row: $\frac{4}{2} = 2$, 9th row: $\frac{6}{6} = 1$, 10th row: n/a

P	x	y	s_1	s_2	s_3		
1	0	$-\frac{1}{6}$	0	$\frac{1}{6}$	0	5	(11) = (7) + (13)
0	0	$\frac{4}{3}$	1	$-\frac{1}{3}$	0	2	(12) = (8) − 2 × (13)
0	0	$-\frac{1}{6}$	0	$\frac{1}{6}$	1	1	(13) = (9) ÷ 6
0	1	$\frac{5}{6}$	0	$\frac{1}{6}$	0	5	(14) = (10) + (13)

Choose y as the pivot column, and apply the ratio test:

12th row: $\frac{2}{\left(\frac{4}{3}\right)} = \frac{3}{2}$, 13th row: n/a , 14th row: $\frac{5}{\left(\frac{5}{6}\right)} = 6$

P	x	y	s_1	s_2	s_3		
1	0	0	$\frac{1}{8}$	$\frac{1}{8}$	0	$\frac{21}{4}$	(15) = (11) + $\frac{1}{6}$ × (16)
0	0	1	$\frac{3}{4}$	$-\frac{1}{4}$	0	$\frac{3}{2}$	(16) = (12) × $\frac{3}{4}$
0	0	0	$\frac{1}{8}$	$\frac{1}{8}$	1	$\frac{5}{4}$	(17) = (13) + $\frac{1}{6}$ × (16)
0	1	0	$-\frac{5}{8}$	$\frac{3}{8}$	0	$\frac{15}{4}$	(18) = (14) − $\frac{5}{6}$ × (16)

The coefficients of s_1 & s_2 in (15) are both positive, so we have maximised P.

Solution: $x = \frac{15}{4}$, $y = \frac{3}{2}$, $P = \frac{21}{4}$ (B)

(3) The Big M (Simplex) Method (same example)

This starts off in the same way as the 2-Stage method, by creating the artifical variable.

$$P - x - y \qquad = 0 \quad (1)$$

$$2x + 3y + s_1 \qquad = 12 \quad (2)$$

$$6x + 5y + \qquad s_2 = 30 \quad (3)$$

$$x + y - s_3 + a_1 = 4 \quad (4)$$

We now modify the objective to:

maximise $P = x + y - Ma_1$, where M is a large number (eg 1000)

This ensures that minimising a_1 is given 1st priority, as the Ma_1 term has the biggest effect on P.

Re-write as $P = x + y - M(4 - x - y + s_3)$

giving $P - (1 + M)x - (1 + M)y + Ms_3 = -4M$

P	x	y	s_1	s_2	s_3	a_1		
1	$-(1+M)$	$-(1+M)$	0	0	M	0	$-4M$	(1)
0	2	3	1	0	0	0	12	(2)
0	6	5	0	1	0	0	30	(3)
0	1	1	0	0	-1	1	4	(4)

We now carry out the Simplex method as usual, and we should find that the RHS of the objective row becomes free of M.

Choose x as the pivot column (eg) and apply the ratio test:

2nd row: $\frac{12}{2} = 6$, 3rd row: $\frac{30}{6} = 5$, 4th row: $\frac{4}{1} = 4$ (as before)

P	x	y	s_1	s_2	s_3	a_1		
1	0	0	0	0	−1	1+M	4	5 = 1 + (1+M)×8
0	0	1	1	0	2	−2	4	6 = 2 − 2×8
0	0	−1	0	1	6	−6	6	7
0	1	1	0	0	−1	1	4	8 = 4

Once M only appears in the a_1 column, we can set a_1 to 0, and remove the a_1 column, arriving at the same tableau as at the end of the 1st stage of the 2-stage method (and then continue as before).

P	x	y	s_1	s_2	s_3		
1	0	0	0	0	−1	4	5
0	0	1	1	0	2	4	6
0	0	−1	0	1	6	6	7
0	1	1	0	0	−1	4	8

(4) Complications for the Simplex Method

The following is a summary of the various ways in which complications can arise.

(i) Objective function parallel to a constraint line (if two variables) or plane (if three).

As for the Linear Programming method, more than one solution is possible.

(ii) Artificial variables may be needed for more than one constraint. In this case, let $A = a_1 + a_2 + \cdots$ for the 2-Stage Simplex, and have $-M(a_1 + a_2 + \cdots)$ in place of $-Ma_1$ for theBig M method.

(iii) When applying the 2-Stage Simplex or Big M method, it may not be possible to reduce A to 0; ie there may not be a solution to the problem.

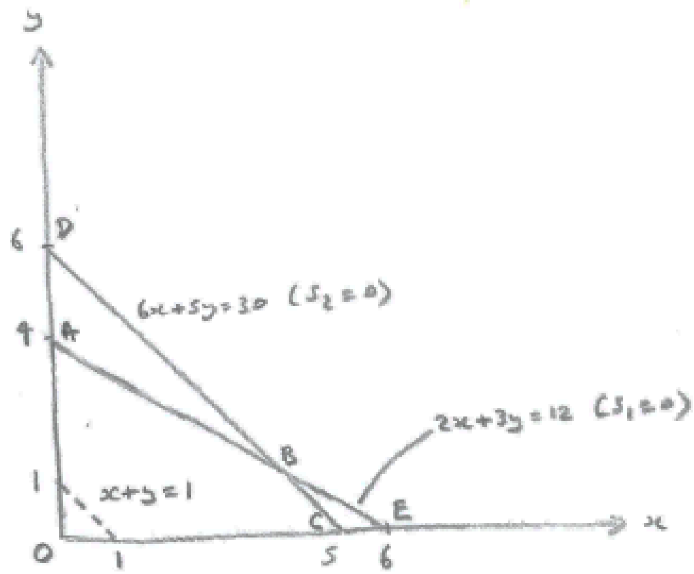
Example (2-Stage Simplex)

Maximise $\mathrm{P} = x + y$

subject to $2x + 3y \geq 12$

$6x + 5y \leq 30$

$y \geq 5$



$P - x - y = 0 \quad (1)$

$2x + 3y - s_1 + a_1 = 12 \quad (2)$

$6x + 5y + s_2 = 30 \quad (3)$

$y - s_3 + a_2 = 5$ (4)

Minimise $A = a_1 + a_2 = (12 - 2x - 3y + s_1) + (5 - y + s_3)$

$\Rightarrow A + 2x + 4y - s_1 - s_3 = 17$

min. max.

A	P	x	y	s_1	s_2	s_3	a_1	a_2		
1	0	2	4	-1	0	-1	0	0	17	(1)
0	1	-1	-1	0	0	0	0	0	0	(2)
0	0	2	3	-1	0	0	1	0	12	(3)
0	0	6	5	0	1	0	0	0	30	(4)
0	0	0	1	0	0	-1	0	1	5	(5)

1st stage

pivot column : y

ratio test : (2) n/a (3) $\frac{12}{3} = 4$ (4) $\frac{30}{5} = 6$ (5) $\frac{5}{1} = 5$

A	P	x	y	s_1	s_2	s_3	a_1	a_2		
1	0	$-\frac{2}{3}$	0	$\frac{1}{3}$	0	-1	$-\frac{4}{3}$	0	1	(6) = (1) − 4×(8)
0	1	$-\frac{1}{3}$	0	$-\frac{1}{3}$	0	0	$\frac{1}{3}$	0	4	(7) = (2) + (8)
0	0	$\frac{2}{3}$	1	$-\frac{1}{3}$	0	0	$\frac{1}{3}$	0	4	(8) = (3) ÷ 3
0	0	$2\frac{2}{3}$	0	$\frac{5}{3}$	1	0	$-\frac{5}{3}$	0	10	(9) = (4) − 5×(8)
0	0	$-\frac{2}{3}$	0	$\frac{1}{3}$	0	-1	$-\frac{1}{3}$	1	1	(10) = (5) − (8)

pivot column : s_1

ratio test : (7) n/a (8) n/a (9) $\frac{10}{(\frac{5}{3})} = 6$ (10) $\frac{1}{(\frac{1}{3})} = 3$

A	P	x	y	s_1	s_2	s_3	a_1	a_2		
1	0	0	0	0	0	0	-1	-1	0	(11) = (6) $-\frac{1}{3}\times$ (15)
0	1	-1	0	0	0	-1	0	1	5	(12) = (7) $+\frac{1}{3}\times$ (15)
0	0	0	1	0	0	-1	0	1	5	(13) = (8) $+\frac{1}{3}\times$ (15)
0	0	6	0	0	1	5	0	-5	5	(14) = (9) $-\frac{5}{3}\times$ (15)
0	0	-2	0	1	0	-3	-1	3	3	(15) = (10) $\times 3$

A has been minimised, with $a_1 = a_2 = 0$

max.

P	x	y	s_1	s_2	s_3		
1	-1	0	0	0	-1	5	(12)
0	0	1	0	0	-1	5	(13)
0	6	0	0	1	5	5	(14)
0	-2	0	1	0	-3	3	(15)

pivot column : x (s_3 is also possible)

ratio test : (14) is the only possible row

P	x	y	s_1	s_2	s_3		
1	0	0	0	$\frac{1}{6}$	$-\frac{1}{6}$	$5\frac{5}{6}$	(16) = (12) + (18)
0	0	1	0	0	-1	5	(17) = (13)
0	1	0	0	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{5}{6}$	(18) = (14) ÷ 6
0	0	0	1	$\frac{1}{3}$	$-\frac{4}{3}$	$4\frac{2}{3}$	(19) = (15) + 2 × (18)

pivot column : s_3

ratio test : (18) is the only possible row

P	x	y	s_1	s_2	s_3		
1	$\frac{1}{5}$	0	0	$\frac{1}{5}$	0	6	(20) = (16) + $\frac{1}{6}$ × (22)
0	$\frac{6}{5}$	1	0	$\frac{1}{5}$	0	6	(21) = (17) + (22)
0	$\frac{6}{5}$	0	0	$\frac{1}{5}$	1	1	(22) = (18) × $\frac{6}{5}$
0	$\frac{12}{5}$	0	1	$\frac{3}{5}$	0	6	(23) = (19) + $\frac{4}{3}$ × (22)

$\Rightarrow$ sol'n is : $x = 0$, $y = 6$ $\quad (s_1 = 6,\ s_2 = 0,\ s_3 = 1)$

$P = 6$

Example (Big M):

Big M method

Maximise $P' = P - (a_1 + a_2)M$

$= x + y - (12 - 2x - 3y + s_1 + 5 - y + s_3)M$

$= (1+2M)x + (1+4M)y - Ms_1 - Ms_3 - 17M$

max.

P'	x	y	s_1	s_2	s_3	a_1	a_2		
1	$-(1+2M)$	$-(1+4M)$	M	0	M	0	0	$-17M$	1
0	2	3	-1	0	0	1	0	12	2
0	6	5	0	1	0	0	0	30	3
0	0	1	0	0	-1	0	1	5	4

pivot row : y

ratio test : (2) $\frac{12}{3} = 4$ (3) $\frac{30}{5} = 6$ (4) $\frac{5}{1} = 5$

P'	x	y	s_1	s_2	s_3	a_1	a_2		
1	$-\frac{1}{3}+\frac{2}{3}M$	0	$-\frac{1}{3}-\frac{1}{3}M$	0	M	$\frac{1}{3}+\frac{4}{3}M$	0	$4-M$	(5) = (1) + (1+4M) × (6)
0	$\frac{2}{3}$	1	$-\frac{1}{3}$	0	0	$\frac{1}{3}$	0	4	(6) = (2) ÷ 3
0	$2\frac{2}{3}$	0	$\frac{5}{3}$	1	0	$-\frac{5}{3}$	0	10	(7) = (3) − 5 × (6)
0	$-\frac{2}{3}$	0	$\frac{1}{3}$	0	−1	$-\frac{1}{3}$	1	1	(8) = (4) − (6)

pivot row : s_1

ratio test : (6) n/a (7) $\frac{10}{(5/3)} = 6$ (8) $\frac{1}{(1/3)} = 3$

P'	x	y	s_1	s_2	s_3	a_1	a_2		
1	−1	0	0	0	−1	M	$1+M$	5	(9) = (5) + $\frac{1}{3}(1+M)$ × (12)
0	0	1	0	0	−1	0	1	5	(10) = (6) + $\frac{1}{3}$ × (12)
0	6	0	0	1	5	0	−5	5	(11) = (7) − $\frac{5}{3}$ × (12)
0	−2	0	1	0	−3	−1	3	3	(12) = (8) × 3

then as per the 2-stage simplex method

(iv) Constraints that are equalities

Replace with two inequality constraints:

ie for $x + y = 4$: replace with $x + y \leq 4$ & $x + y \geq 4$

Example

Big M method

Maximise $x+y$, subject to $2x+y \leq 12$ and $x=4$

$2x + y + s_1 = 12$

$x - s_2 + a_1 = 4$

$x + s_3 = 4$

Modified objective: maximise $x + y - Ma_1$ (P)

$= x + y - M(4 - x + s_2)$

$= (1+M)x + y - Ms_2 - 4M$

p	x	y	s_1	s_2	s_3	a_1		
1	$-(1+M)$	-1	0	M	0	0	$-4M$	①
0	2	1	1	0	0	0	12	②
0	1	0	0	-1	0	1	4	③
0	1	0	0	0	1	0	4	④

pivot row : x

ratio test : ② $\frac{12}{2} = 6$ ③ $\underline{\frac{4}{1} = 4}$ ④ $\frac{4}{1} = 4$

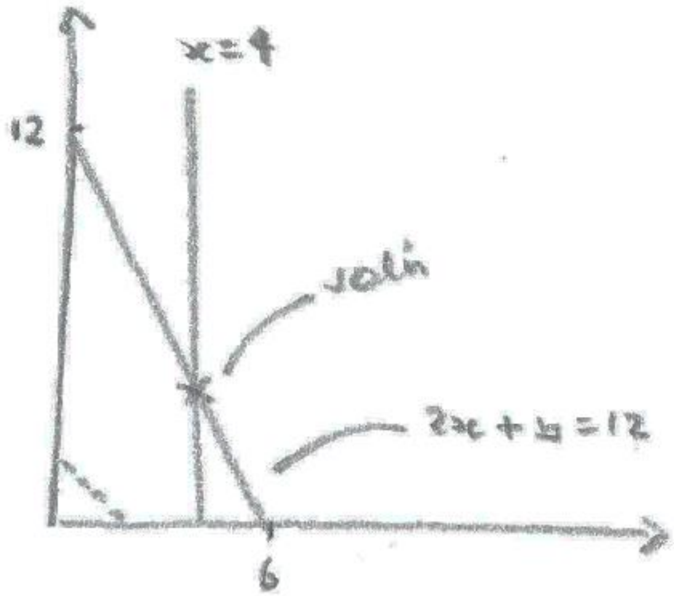
p	x	y	s_1	s_2	s_3	a_1		
1	0	-1	0	-1	0	$1+M$	4	⑤ = ① + (1+M) × ⑦
0	0	1	1	2	0	-2	4	⑥ = ② − 2 × ⑦
0	1	0	0	-1	0	1	4	⑦ = ③
0	0	0	0	1	1	-1	0	⑧ = ④ − ⑦

pivot col : y ⇒ pivot row : ⑥

p	x	y	s_1	s_2	s_3		
1	0	0	1	1	0	8	⑨ = ⑤ + ⑩
0	0	1	1	2	0	4	⑩ = ⑥
0	1	0	0	-1	0	4	⑪ = ⑦
0	0	0	0	1	1	0	⑫ = ⑧

⇒ sol'n is : $x = 4$, $y = 4$ ($s_1 = 0$, $s_2 = 0$, $s_3 = 0$)

$P = 8$



Note: If there is a constraint such as, for example:

$x + y + z = 100$, then this can enable the variable z (for example) to be eliminated from the problem (noting that the constraint

$z \geq 0$ becomes the constraint $100 - x - y \geq 0$ or $x + y \leq 100$).

This can enable a 3-variable problem to be tackled by a graphical method (involving a feasible region), rather than having to employ the Simplex method.

(v) $x + y < 4$ (eg)

Use $x + y \leq 4$ instead, and reduce x *or* y slightly, if necessary.

(vi) Big M method: to minimise $P = x + y$

Modify to minimising $x + y + Ma_1$ (instead of maximising

$x + y - Ma_1$)

(vii) If x (for example) can be negative, then replace x with $x_1 - x_2$, where $x_1, x_2 \geq 0$ (This allows x to be negative, if necessary.)