TMUA 2022 Paper 2 Solutions (10 pages; 7/8/24)

Q1

$\frac{dy}{dx} = 0 \Rightarrow 12x^3 + 12x^2 + 12x = 0$

$\Rightarrow x(x^2 + x + 1) = 0$

As the discriminant of $x^2 + x + 1$ is negative, there are no roots of $x^2 + x + 1 = 0$, and so there is just 1 stationary point , when $x = 0$.

Answer: B

Q2

Coeff. of x^0 in $(1 + x)^5$

+ Coeff. of x^1 in $(1 + x)^5$

+ … Coeff. of x^5 in $(1 + x)^5$

= sum of Binomial coeffs of $(1 + x)^5$

$= (1 + 1)^5 = 32$

Answer: E

Q3

I: Not a counter-example, as 2 is prime and $2^2 + 2$ is not prime.

II: This is a counter-example, as 3 is prime but $3^2 + 2$ is prime.

III: Not a counter-example, as 4 is not prime.

Answer: C

Q4

The eq'n of the circle can be rewritten as

$(x+f)^2+(x+g)^2+\cdots=0$

The centre is then $(-f,-g)$.

Answer: B

Q5

[The converse of $X \Rightarrow Y$ is $Y \Rightarrow$ X (or $X \Leftarrow Y$).

The contrapositive of $X \Rightarrow Y$ is $Y' \Rightarrow$ X$'$. This is mathematically equivalent to $X \Rightarrow Y$.]

I: Yes

II: No

III: Yes

Answer: F

Q6

Examples: (a) 2 5 5 7 8 (b) 2 5 6 7 (c) 2 5 5 7

A: False: P is sufficient for Q, but not necessary – as shown by (c)

B: False: again, P is not necessary for Q (also, it is sufficient for Q).

C: True (as mentioned under (A))

D: False: P is not necessary for Q, but it is sufficient

Answer: C

Q7

Consider the simpler example $f(x) = x^2 + 1$: This cannot be factorised, but $f(3) = 10$ CAN be factorised. (Any factorisation of a quadratic has to hold for all values of x, but

$x^2 + 1 = (x - 1)(x + 2)$ when $x = 3$)

Answer: F

Q8

The possible pairs where the sum is 74 are:

70,4; 67,7; ... 40,34

$70 = 1 + 3 \times 23$ & $40 = 1 + 3 \times 13$

So there are 11 such pairs.

If (*) doesn't hold, at least one of each of these pairs has to be excluded.

There are 24 numbers in the sequence ($1 = 1 + 3 \times 0$ up to $1 + 3 \times 23$), and at least 11 have to be excluded, in order for (*) not to hold. This leaves 13.

So, if $n = 14$, then it isn't possible to exclude 11 (but if $n = 13$ then it is possible to exclude 11).

Answer: C

Q9

Considering the graph of $y = x^2$, a sufficiently large negative value of x will result in x^2 exceeding any chosen value.

Answer: A

Q10

I: False

II: True

III: True (eg $x = 0$)

Answer: G

Q11

[It is usually a good idea to look ahead in the question (or the multiple choice options) for ideas. Also, problems can generally be tackled by either starting with something that is given (eg one of the options given here), or by considering (say) a standard result (such as Pythagoras' theorem).]

To start with, consider $PQ^2 + PS^2 = (x^2 + y^2) + (x^2 + z^2)$

This will equal $SQ^2 = (z + y)^2$ when $2x^2 = 2zy$

So C ensures that $PQ^2 + PS^2 = SQ^2$, and by the converse of Pythagoras' theorem, C is therefore a sufficient condition for angle SPQ to be a right angle. And by Pythagoras' theorem itself, it is also a necessary condition.

Thus we have established that C is the answer.

[As a check:

A: Sufficient, but not necessary. (Consider an example where x is very short compared with z.)

B: This is equivalent to $x = \frac{1}{2}(y + z)$

Consider again an example of a right angled triangle SPQ where x is very short compared with z, such that $x < \frac{1}{2}(y + z)$.

Thus, B is not a necessary condition.

D: Not sufficient (consider large x – compared with y & z)

E : Not necessary (consider the case A, where $x = y = z$, which is sufficient for angle SPQ to be a right angle, but doesn't satisfy E)]

Answer: C

Q12

$(\sqrt{2})^x = 2^{x/2}$

For $0 < x < 1, \sqrt{x} > x > \frac{x}{2}$, so that $2^{\sqrt{x}} > 2^x > 2^{x/2}$ (as $y = 2^x$ is an increasing function).

So $P > Q > R$.

Answer: F

Q13

Experimenting with different values of x, we see that a problem only occurs when $x = 1$.

Answer: E

Q14

The graph of $y = |x + a|$ is obtained by translating the graph of

$y = |x|$ by a to the left. The graphs of $y = |x + 11|$ and

$y = |x + 5|$ intersect halfway between $x = -11$ & $x = -5$;

ie at $x = -8$; so that $|x + 5| < |x + 11|$ when $x > -8$

Similarly, the graphs of $y = |x + 11|$ and $y = |x + 1|$ intersect

halfway between $x = -11$ & $x = -1$; ie at $x = -6$;

so that $|x + 11| < |x + 1|$ when $x < -6$

Hence both inequalities are satisfied when $-8 < x < -6$

Answer: D

Q15

$log_x y \,.\, log_y z = log_x z$ (*)

Also $log_z x = \frac{1}{log_x z} = \frac{1}{log_x y \,.log_y z} = \frac{1}{zx}$

Answer: F

Q16

[Experimenting:]

Without loss of generality the a_n can be taken to be in increasing order; eg 3,4,5, …

Consider the case where the b_n are $1, 2, 3, \ldots$

and the c_n are $4, 5, 6, \ldots$

As it stands, statement I holds, but we can change some of the terms in the b_n & c_n sequences as follows, so that it no longer holds:

Let the b_n now be $10, 1, 3, \ldots$, and the c_n be $1, 10, 6, \ldots$

So statement I need not be true.

Also, statement II need not be true, as it isn't true for the 1st example above.

For statement III,

$max(a_i) = a_p$, for some p

Then, $a_p \leq b_p + c_p \leq b_q + c_r$,

where $max(b_i) = b_q$ and $max(c_i) = c_r$

So $max(a_i) \leq max(b_i) + max(c_i)$;

ie statement III is true.

Answer: D

Q17

E is the correct answer. The solution could perhaps be made a bit clearer by writing:

"IV Hence, from II and III …"

Answer: E

Q18

When $x = 0$:

$cosx^{cosx} = 1, sinx^{sinx}$ is undefined,

$cosx^{sinx} = 1\ , sinx^{cosx} = 0$

So Q must be $sinx^{cosx}$

When $x = \frac{\pi}{2}$:

$cosx^{cosx} =$ is undefined, $sinx^{sinx} = 1$

$cosx^{sinx} = 0\ , sinx^{cosx} = 1$

So P must be $cosx^{sinx}$

[The fact that $sinx^{sinx}$ is undefined at $x = 0$, and that $cosx^{cosx}$ is undefined at $x = \frac{\pi}{2}$, strongly suggests that $sinx^{sinx}$ is S – where the graph steeply approaches 1 as $x \to 0^+$ (and similarly, $cosx^{cosx}$ is R).]

When $x = \frac{\pi}{6}$, $sinx^{sinx} = (\frac{1}{2})^{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$,

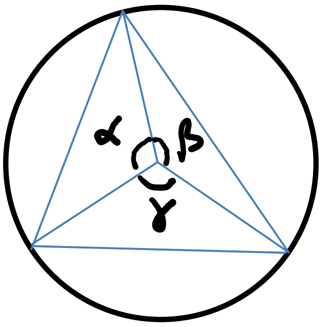
and as $0.7 < \frac{\sqrt{2}}{2} < 0.8$, this rules out R.

So $cosx^{cosx}$ must be R, $sinx^{sinx}$ must be S,

$cosx^{sinx}$ must be P, and $sinx^{cosx}$ must be Q

Answer: E

Q19



Considering the case $n = 3$ (see diagram), equal areas ⇒

$\frac{1}{2}r^2 sin\alpha = \frac{1}{2}r^2 sin\beta = \frac{1}{2}r^2 sin\gamma$, where r is the radius of the circle.

So $sin\alpha = sin\beta = sin\gamma$ (and $\alpha + \beta + \gamma = 360$).

As the angles are all between 0 & 180 (degrees), either $\beta = \alpha$

or $\beta = 180 - \alpha$, and similarly for the other two pairs.

Suppose then that $\beta = 180 - \alpha$ (or similarly for one of the other pairs). Then $\alpha + \beta + \gamma = 360 \Rightarrow 180 + \gamma = 360 \Rightarrow \gamma = 180$,

which isn't possible. So it follows that $\beta = 180 - \alpha$ etc isn't possible, and hence $\alpha = \beta = \gamma$ is the only possibility.

Then the 3 triangles are congruent, and so the sides of the polygon are equal, and the polygon is therefore regular.

So the given properties are sufficient to deduce that the polygon is regular when $n = 3$.

Consider now the case when $n = 4$.

In that case we have $sin\alpha = sin\beta = sin\gamma = sin\delta$

and $\alpha + \beta + \gamma + \delta = 360$

A possible solution is $\alpha = 60, \beta = 120, \gamma = 60, \delta = 120$, so that the given properties are not sufficient to deduce that the polygon is regular when $n = 4$.

The same logic applies to higher values of n. For example, when $n = 5$, we could set $\gamma = \delta = \varepsilon = \frac{180}{3} = 60$, with $\alpha = 60$ & $\beta = 120$

Thus the given properties are only sufficient to deduce that the polygon is regular when $n = 3$.

Answer: B

Q20

$f_2(x)$ has a range of $[sin(-1), sin(1)]$,

and so $max\{f_2(x)\} = sin(1) < sin\left(\frac{\pi}{2}\right) = 1$;

then $f_3(x)$ has a range of $[cos(sin(1)),1]$

and so $max\{f_3(x)\} = 1$

Thus A & B can be ruled out.

Also, $0 < cos\big(sin(1)\big) < 1$, so that

$f_4(x)$ has a range of $[a, b]$, where $0 < a < b < 1$

So $max\{f_4(x)\} < 1$,

and $f_5(x)$ has a range of $[c, d]$, where $0 < c < d < 1$

So $max\{f_5(x)\} < 1$ also.

Thus C& D can be ruled out.

Then $max\{f_4(x)\} = \sin\,(\cos[\min|f_2(x)|])$

$= \sin(\cos(0)) = \sin(1) = max\{f_2(x)\}$

Answer: E