TMUA 2023 Paper 2 Solutions (13 pages; 11/9/24)

Q1

$\frac{1}{\sqrt{x}-6} - \frac{1}{\sqrt{x}+6} = \frac{3}{11}$

$\Rightarrow \frac{12}{x-36} = \frac{3}{11} = \frac{12}{44}$

$\Rightarrow x - 36 = 44$ & so $x = 80$

Answer : H

Q2

Integral $= \int_9^{16} 4\, dx = [4x]_9^{16} = 4(16-9) = 28$

Answer : F

Q3

Answer : C

Q4

VI is incorrect : 3 is a multiple of 3, but is prime

Answer : G

Q5

If R is true, then$\int_0^k sin2x\,dx = [-\frac{1}{2}cos2x]_0^k$

$= -\frac{1}{2}(1-1) = 0$; so $R \Rightarrow S$

If S is true, then $\left[-\frac{1}{2}cos2x\right]_0^k = 0$,

so that $(cos2k) - 1 = 0$,

and hence $2k$ is an integer multiple of 2π; ie R is true

Answer : A

Q6

$a^x = x$ is equivalent to $log_a x = x$, so I has the same number of sol'ns as (*)

$a^x = x \Rightarrow a^{2x} = x^2$, but $a^{2x} = x^2 \Rightarrow a^x = x \; or \; a^x = -x$

[so potentially there may be extra sol'ns to II where $x < 0$]

Consider $a = 2$, and let $y = 2^x + x$. We want to see if there are

any (negative) values of x for which $y = 0$:

When $x = -1, y = -\frac{1}{2} < 0$, and when $x = -\frac{1}{2}$, $y = \frac{1}{\sqrt{2}} - \frac{1}{2}$

$= \frac{\sqrt{2}-1}{2} > 0$

So, as $2^x + x$ is a continuous function, the change of sign means

That there is a solution to $2^x + x = 0$, and thus to $a^x = -x$.

So II has more sol'ns than (*).

Finally, there is a 1-1 correspondence between sol'ns of $a^x = x$

and sol'ns of $a^{2x} = 2x$ (setting $y = 2x$), so that III has the same number of sol'ns as (*).

Answer : F

Q7

For $ax + by = c$, a positive gradient means that $-\frac{a}{b} > 0$, or $\frac{a}{b} < 0$,

and a positive y-intercept means that $\frac{c}{b} > 0$

Let (*) be the situation where both the gradient and the y-intercept are positive.

Thus, A is equivalent to (*); ie A is a necessary and sufficient condition for (*).

B: $\frac{a}{b} > 0$ is not a necessary condition

C: Let $a = -1, b = 2$ & $c = 3$, so that

$\frac{a}{b} = -\frac{1}{2} < 0$ and $\frac{c}{b} = \frac{3}{2} > 0$, and hence (*) is satisfied

But it is not true that $a > b > c$, so that this is not a necessary condition for (*).

D: Now let $a = -1, b = 3$ & $c = 2$, so that

$\frac{a}{b} = -\frac{1}{3} < 0$ and $\frac{c}{b} = \frac{2}{3} > 0$, and hence (*) is satisfied

But it is not true that $a < b < c$, so that this is not a necessary condition for (*).

For E & F: Suppose that $b > 0$. Then (*) $\Rightarrow \frac{a}{b} < 0 \Rightarrow a < 0$

and $\frac{c}{b} > 0 \Rightarrow c > 0$

If instead $b < 0$. Then (*) $\Rightarrow \frac{a}{b} < 0 \Rightarrow a > 0$

and $\frac{c}{b} > 0 \Rightarrow c < 0$

Thus (as $b \neq 0$), a & c must be of opposite sign; ie this is a necessary condition.

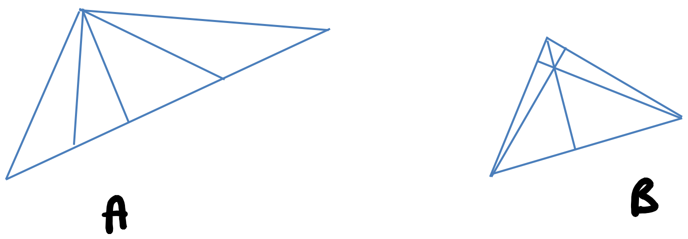
So F can be ruled out.

By elimination, we can conclude that E is the correct answer.

[E is not sufficient: consider the case $a = 1, b = 1, c = -1$, where $\frac{a}{b} > 0$, so that (*) is not satisfied.]

Answer : E

Q8



In the case of an obtuse-angled triangle (as in A), III will always hold. In the case of an acute-angled triangle (as in B), III will again always hold. [The 'altitudes' of an acute-angled triangle always meet at a point (the 'othocentre') inside the triangle.]

Answer : D

Q9

Statement (*) is clearly not true (the sum of the angles in a pentagon is $(5-2)\times 180 = 540°$, and so the other angles need only average 108°). The contrapositive of (*) is mathematically equivalent to (*), and so is also not true. [The contrapositive of $A \Rightarrow B$ is $B' \Rightarrow A'$]

The converse of (*) is: "If the interior angle form an arithmetic sequence, then at least one of them is 108°"

To investigate this, suppose that the smallest angle is α, and that the common difference of the sequence is d.

Then $5 \times \frac{1}{2}(\alpha + [\alpha + 4d]) = 5(\alpha + 2d) = 540$,

so that $\alpha + 2d = 108$, and thus one of the angles is 108°.

So the converse of (*) is true.

Answer : D

Q10

Answer : A

Q11

Writing A is the event that $2^k + 1$ is prime, and B is the event that

k is a power of 2:

$(*) \equiv A \Rightarrow B$

$I \equiv B \Rightarrow A$ (the converse of (*)), and so $I \not\equiv (*)$

$II \equiv A' \Rightarrow B' \equiv B \Rightarrow A$ [$A' \Rightarrow B'$ is the contrapositive *of*

$B \Rightarrow A$, and so $II \not\equiv (*)$

$III \equiv A \Rightarrow B$, and so $III \equiv (*)$

Answer : G

Q12

Statement I means that:

"If $sinxcos^2x = p^2sinx$ has 3 solutions, then $p > 1$"

Now, $sinxcos^2x = p^2sinx \Rightarrow sinx(p^2 - cos^2x) = 0$

$x = 0, \pi$ & 2π are 3 solutions

Suppose now that $p^2 - cos^2x = 0$. Then $cosx = \pm p$.

This gives rise to further solutions, unless $p > 1$ or $p < -1$

(when $p = \pm 1, x = 0, \pi\ or\ 2\pi$)

So statement I is not true.

Statement II means that:

"If $sinxcos^2x = p^2sinx$ has 7 solutions, then $-1 < p < 1$"

["Only if" is equivalent to "is sufficient for"]

There will be 7 solutions when $p^2 - cos^2x = 0$ (*) has 4 solutions, other than $x = 0, \pi\ or\ 2\pi$

As before, when $p = \pm 1$, (*) $\Rightarrow x = 0, \pi\ or\ 2\pi$.

When $p < -1\ or\ p > 1$, (*) has no solutions.

And when $-1 < p < 1$, (*), $cosx = \pm p$, which gives 4 solutions (not including $x = 0, \pi\ or\ 2\pi$).

So statement II is true.

Answer : C

Q13

[It may be worth starting at C, as the examiners could well be expecting most candidates to start at A, or E!]

$C \Rightarrow A, C \nRightarrow B, C \Rightarrow D$ & $C \Rightarrow E$ so C is the correct answer

[$A \nRightarrow B$ & $A \nRightarrow C$, so A is ruled out]

[$B \Rightarrow A, B \Rightarrow C, B \Rightarrow D$ & $B \Rightarrow E$, so B is not the answer]

[$D \nRightarrow A, D \nRightarrow B$, so D is ruled out]

[$E \Rightarrow A, E \nRightarrow B, E \nRightarrow C$, so E is ruled out]

Answer : C

Q14

It is assumed that the wording means “If ANY two of the lines …”

[Always assume the simplest interpretation.]

Suppose that the 1st two lines are parallel. Then $-\frac{a}{b} = -\frac{b}{c}$ (1)

The gradient of the 3rd is $-\frac{c}{a} = -\frac{b^2}{a}$

If the 3 gradients are the same, then $-\frac{b^2}{a} = -\frac{a}{b}$,
so that $b^3 = a^2$.

But (1) can be satisfied without $b^3 = a^2$ holding, so A is not correct.

If the 3rd line is perpendicular to the other two, then

$(-\frac{c}{a})(-\frac{a}{b}) = -1$; ie $c = -b$

Again, (1) can be satisfied without $c = -b$ holding, so B is not correct.

If the 3rd line is parallel to $y = x$, then $-\frac{c}{a} = 1$, so that $c = -a$

Once again, (1) can be satisfied without $c = -a$ holding, so C is not correct.

If the 3rd line is perpendicular to $y = x$, then $-\frac{c}{a} = -1$, so that $c = a$

And again, (1) can be satisfied without $c = a$ holding, so D is not correct.

Suppose that the 1st two lines are perpendicular. Then $\left(-\frac{a}{b}\right)\left(-\frac{b}{c}\right) = -1$; ie $c = -a$ (2)

[Leave E for the moment, as it is more complicated to investigate.]

As before, if the 3rd line is parallel to $y = x$, then $-\frac{c}{a} = 1$, so that $c = -a$, which is consistent with (2), and if $c = -a$ then the 3rd line is parallel to $y = x$.
[We have to be careful to show that the 3rd line being parallel is a necessary condition, rather than a sufficient condition.]

If instead the 1st and 3rd lines are perpendicular, then $\left(-\frac{a}{b}\right)\left(-\frac{c}{a}\right) = -1$; ie $c = -b$, and the gradient of the 2nd line is

$-\frac{b}{c} = 1$, so that the 2nd line is parallel to $y = x$.

And if instead the 2nd and 3rd lines are perpendicular, then $\left(-\frac{b}{c}\right)\left(-\frac{c}{a}\right) = -1$; ie $b = -a$, and the gradient of the 1st line is

$-\frac{a}{b} = 1$, so that the 1st line is parallel to $y = x$.

Hence F is correct.

Answer : F

Q15

$0.00110011\ ... = (2^{-3} + 2^{-4}) + (2^{-7} + 2^{-8}) + (2^{-11} + 2^{-12})\ ...$

$= (2^{-3} + 2^{-4})(1 + 2^{-4} + 2^{-8} + \cdots)$

$= \frac{1}{8}\left(1 + \frac{1}{2}\right)\frac{1}{1-2^{-4}} = \frac{3}{16} . \frac{16}{15} = \frac{1}{5}$

Answer : B

Q16

As 7 is a divisor of $u_1 = a\ \&\ u_2 = b$, it is a divisor of $u_3 = u_1 + u_2$, and of $u_4 = u_2 + u_3$ etc; so that 7 is a divisor of u_{2023};

ie statement I is true

$A \Rightarrow B$ is equivalent to $B' \Rightarrow A'$ [consider Venn diagram, where $A \subset B$]

So Statement II is equivalent to:

“If u_1 is a factor of u_n for some $n > 1$,

then u_1 is a factor of u_2” (*)

Consider $u_1 = 2, u_2$=3; so that $u_3 = 5, u_4 = 8$

Then (*) doesn’t hold (and we can also see directly that Statement II isn’t true).

For Statement III:

Now the HCF of a & b is 7, so that $a = 7m$ & $b = 7n$, where the HCF of the positive integers m & n is 1. Then $u_3 = a + b$,

$u_4 = (a + b) + b$ and $u_5 = (a + 2b) + (a + b) = 2a + 3b$

[On account of the presence of $3b$ here:]

Consider the case of $a = 21$ & $b = 7$. Then $u_5 = 2(21) + 3(7)$

$= 21(2 + 1)$, and as u_1 & u_5 have a HCF of 21, this provides a counterexample to Statement III, which is therefore not true.

Thus only Statement I is true.

Answer : B

Q17

Value of integral is area under the 'curve':

$2^1.1 + 2^2.1 + \cdots + 2^{99}.1$

$= 2.\frac{2^{99}-1}{2-1} = 2^{100} - 2$

Answer : F

Q18

If $b^2 > 4c$, then the equation $y^2 + by + c = 0$ (*) has the distinct roots $y = \frac{-b \pm \sqrt{b^2-4c}}{2}$

Then, in order for the equation $x^4 + bx^2 + c = 0$ to have 4 distinct roots, both of the roots of (*) must be positive, so that

$-b - \sqrt{b^2 - 4c} > 0$; ie $\sqrt{b^2 - 4c} < -b$, which requires $b < 0$ and $c > 0$

Thus, sufficient and necessary conditions for 4 distinct roots are:

$b^2 > 4c, b < 0$ and $c > 0$,

or equivalently $c > 0, b < -2\sqrt{c}$

Answer : D

Q19

Consider $f(x) = x^2 + x - 1$, so that $f(x) = 0$ has 2 distinct roots.

Then $g(x) = x(2x + 1) = 2x^2 + x$, so that $g(x) = 0$ has 2 distinct roots; ie $M = N$

Thus Statement II is true.

Instead, let $f(x) = x^2 + x + 1$, so that $f(x) = 0$ has no roots.

Then $g(x) = x(2x + 1) = 2x^2 + x$ again, so that $g(x) = 0$ has 2 distinct roots; ie $M < N$

Thus Statement I is true.

[If $f(x)$ is a quadratic, then we can see from the above that $g(x) = 0$ will always have 2 distinct roots; thus $M > N$ isn't possible in this situation.]

Considering the graph of $y = f(x)$, we see that, for there to be M roots, there must be at least $M - 1$ turning points. [Consider a cubic, for example.] But one of these turning points could occur when $x = 0$. For example, if $f(x) = x^2(x - 1) + a$, where $a > 0$ is sufficiently small for the graph of $y = f(x)$ to cross the x-axis 3 times.

Then $g(x) = x[2x(x-1) + x^2] = x^2(3x-2)$. So $M = 3$

and $N = 2$

Thus Statement III is also true.

Answer : H

Q20

"Only if" is equivalent to "implies".

Let $f(x) = x^2$. Then $f(|x|) = f(x)$ for all x, and so the integrand of $I_{p,q}$ is zero for all x (and hence $I_{p,q} = 0$), regardless of the value of p. Thus $I_{p,q} = 0 \nRightarrow 0 < p$, and so Statement I is not true.

[It looks as if it would be difficult to prove the truth of Statement II, so it may be best to look for a counterexample first.]

Consider $f(x) = -x$ (so that $f'(x) < 0$ for all x).

Then the integrand of $I_{p,q}$ is $(-x)^2 - (-|x|)^2 = 0$ for all x, and hence $I_{p,q} = 0$. Thus Statement II is not true.

[If we were short of time, this would be a good one to guess!]

Re. Statement III, if $p \geq 0$, then $f(|x|) = f(x)$, and so the integrand of $I_{p,q}$ is zero, and therefore $I_{p,q} = 0$. Thus if $I_{p,q} > 0$, it follows that $p < 0$.

Thus Statement III is true.

[Alternatively, we can say that Statement III ($I_{p,q} > 0 \Rightarrow p < 0$) is equivalent to "$not\ (p < 0) \Rightarrow not(I_{p,q} > 0)$", or

$p \geq 0 \Rightarrow I_{p,q} \leq 0$ (*)

We have shown that $p \geq 0 \Rightarrow I_{p,q} = 0$, which means that (*) is true.]

Answer : D