

Logarithms (TMUA) – Ideas & Exercises (10 pages; 14/12/24)

Rewriting logarithm equation as exponential equation:

$$\log_a b = c \Leftrightarrow a^c = b$$

Write $3 + 2\log_2 5$ in the form $\log_a b$

$$\begin{aligned}3 + 2\log_2 5 &= 3\log_2 2 + \log_2(5^2) \\ &= \log_2(2^3) + \log_2(5^2) = \log_2(8 \times 25) = \log_2(200)\end{aligned}$$

$$\log_a b \log_b c = \log_a c \quad \text{or} \quad \log_b c = \frac{\log_a c}{\log_a b}$$

Prove this result

$$\log_a b \log_b c = \log_a c \quad \text{or} \quad \log_b c = \frac{\log_a c}{\log_a b}$$

Solution

rtp $\log_a b \log_b c = \log_a c$ (*)

Method 1

Let $b = a^x$ & $c = b^y$

Then $c = (a^x)^y = a^{xy}$

and $\log_a c = xy = \log_a b \log_b c$, as required

Method 2

(*) is equivalent to $a^{\log_a b \log_b c} = a^{\log_a c}$ (as $y = a^x$ is an increasing function)

ie $(a^{\log_a b})^{\log_b c} = c$ (**)

and the LHS equals $b^{\log_b c} = c$, so that (**) holds, and hence (*) holds also

Relation connecting $\log_x y$ and $\log_y x$?

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$$\log_b c = \frac{\log_a c}{\log_a b}$$

Writing $b = x, c = y$ & $a = y$: $\log_x y = \frac{1}{\log_y x}$

Use linear interpolation to show that $\log_8 36 > \frac{3}{2}$

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As $\log_8 8 = 1$ and $\log_8 64 = 2$, and as $y = \log_8 x$ is a concave function (ie its gradient is always decreasing, so that $\frac{d^2y}{dx^2} < 0$)

linear interpolation $\Rightarrow \log_8 \left[\frac{1}{2}(8 + 64) \right] > \frac{1}{2}(1 + 2)$

ie $\log_8 36 > \frac{3}{2}$

Find an upper bound of the form $\frac{m}{n}$ for $\log_2 3$

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Suppose that $\log_2 3 < \frac{m}{n}$

Then $3 < 2^{\left(\frac{m}{n}\right)}$ and $3^n < 2^m$

As $243 = 3^5 < 2^8 = 256$, $\log_2 3 < \frac{8}{5}$

[and $\frac{8}{5}$ is a reasonably low upper bound, as 243 & 256 are reasonably close]

Given that $\log_2 3 < \frac{8}{5}$, show that $\log_2 12 < \frac{18}{5}$

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$$\log_2 12 = \log_2(3 \times 4) = \log_2 3 + \log_2 4 < \frac{8}{5} + 2 = \frac{18}{5},$$

Show that $\log_5 10 < \frac{3}{2}$

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$$\log_5 10 < \frac{3}{2} \Leftrightarrow 10 < 5^{\left(\frac{3}{2}\right)} \text{ (as the log function is increasing)}$$

$$\Leftrightarrow 10^2 < 5^3 \Leftrightarrow 100 < 125$$

What relation is there between $\log_2 a$ and $\log_8 a$?

[Idea: Consider relation between powers.]

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Solution

Method 1

$$\log_2 a \times \log_a 8 = \log_2 8 = 3,$$

$$\text{so that } \log_2 a = \frac{3}{\log_a 8} = 3\log_8 a$$

[Standard result: $\log_a b \times \log_b a = \log_a a = 1$,

$$\text{so that } \log_b a = \frac{1}{\log_a b}]$$

Method 2

$$(2^3)^p = 2^{3p} = a \text{ (say)}$$

$$\text{so that } \log_8 a = \frac{1}{3} \log_2 a$$

Show that $\log(4 - \sqrt{15}) = -\log(4 + \sqrt{15})$

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Solution

$$\log(4 - \sqrt{15}) = -\log\left(\frac{1}{4 - \sqrt{15}}\right) = -\log\left(\frac{4 + \sqrt{15}}{16 - 15}\right) = -\log(4 + \sqrt{15})$$

[or consider $\log(4 - \sqrt{15}) + \log(4 + \sqrt{15})$]