Vectors: Exercises - Overview (3/5/24)

Lines

Q5

Given that the line $\underline{r} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ can also be written as

$\begin{pmatrix} 0 \\ 7 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, find μ in terms of λ

Q6

Find a vector equation of the line that passes through the point (1,2) and is perpendicular to the line $\underline{r} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

Lines & Planes

Q1

(i) Show that the line $\underline{r} = \underline{a} + t\underline{b}$ and the plane $\underline{r}.\underline{n} = d$ intersect at the point $\underline{r} = \underline{a} + \left(\frac{d - \underline{a}.\underline{n}}{\underline{b}.\underline{n}}\right)\underline{b}$

(ii) Find the intersection of the line $\underline{r} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and the plane $\underline{r}.\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = -2$

(iii) Find the angle between the line and the plane in (ii).

Q8

Find the acute angle between the line $\frac{x-4}{-3} = \frac{y+2}{5}$, $z = -2$ and the plane $2x - z = 7$.

Q10

(i)(a) Find the acute angle between the line $\frac{\boldsymbol{x}}{\mathbf{2}} = \frac{\boldsymbol{y}+\mathbf{1}}{-\mathbf{3}} = \frac{\boldsymbol{z}-\mathbf{2}}{\mathbf{1}}$ and the plane $\boldsymbol{x} + \boldsymbol{y} - \mathbf{2}\boldsymbol{z} = \mathbf{5}$

(b) Show that the same angle is obtained if the line is written in the form

$\frac{\boldsymbol{x}}{-\mathbf{2}} = \frac{\boldsymbol{y}+\mathbf{1}}{\mathbf{3}} = \frac{\boldsymbol{z}-\mathbf{2}}{-\mathbf{1}}$ (ie without rearranging into the form in (a))

(ii)(a) Find the acute angle between the planes $\boldsymbol{x} + \mathbf{4}\boldsymbol{y} - \mathbf{3}\boldsymbol{z} = \mathbf{7}$

and $\boldsymbol{x} - \boldsymbol{y} + \mathbf{4}\boldsymbol{z} = \mathbf{2}$

(b) Find the acute angle between the planes $\boldsymbol{x} + \mathbf{4}\boldsymbol{y} - \mathbf{3}\boldsymbol{z} = \mathbf{7}$ and
$-\boldsymbol{x} + \boldsymbol{y} - \mathbf{4}\boldsymbol{z} = \mathbf{2}$ (again, without rearranging the equation)

Q17

Find the line that is the reflection of the line $\frac{\boldsymbol{x}-\mathbf{2}}{\mathbf{3}} = \frac{\boldsymbol{y}}{\mathbf{4}} = \frac{\boldsymbol{z}+\mathbf{1}}{\mathbf{1}}$ in the plane $\boldsymbol{x} - \mathbf{2}\boldsymbol{y} + \boldsymbol{z} = \mathbf{4}$

Q20

Find the reflection of the line $\frac{x-2}{3} = \frac{y+4}{1}$; $z = 3$ in the plane $y = 4$

Planes

Q7

Find the angle between the planes $x = 2$ and $y + 2z = 3$

Q9

Find the cartesian form of the plane

$$\underline{r} = \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix} + s\begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

Q11

Find the plane containing the points
$(2, -1,4), (-3,4,2)$ and $(1,0,5)$, in Cartesian form

Q15

(i) Find a vector that is perpendicular to both $\begin{pmatrix} 7 \\ 0 \\ -10 \end{pmatrix}$ & $\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$

(ii) Use (i) to find the plane that passes through the points with

position vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 8 \\ 2 \\ -7 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}$

Q25

(i) Find the plane containing the points
$(3,0, -1), (5,2, -3)$ and $(4,2,4)$, in parametric form

(ii) Hence find the equation of the plane in Cartesian form.

Q26

Write the following Cartesian equations of planes in parametric form:

(i) $x + y + z = 1$ (ii) $x + y = 1$ (iii) $x = 1$

Shortest Distance

Q2

Find the shortest distance between the lines

$\frac{x-2}{4} = \frac{y-1}{3} = \frac{z+3}{2}$ and $\frac{x+5}{7} = \frac{y}{1} = \frac{z-1}{3}$

Q3

Find the shortest distance between the lines

$\frac{x-1}{2} = \frac{y+3}{5} = \frac{z-2}{3}$ and $\frac{x}{1} = \frac{y-4}{2} = \frac{z+1}{2}$, identifying the points on the lines at which this shortest distance occurs.

Q12

(i) Find the intersection of the line $\underline{r} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ and the plane $3x + y + 4z = 77$

(ii) Find the shortest distance from the point $\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$ to the plane

$3x + y + 4z = 77$

Q13

(i) Given that the shortest distance from the point $\underline{p}$ to the plane $\underline{r}.\underline{n} = d$ is $\frac{|d-\underline{p}.\underline{n}|}{|\underline{n}|}$, what is the significance of $\frac{d}{|\underline{n}|}$ if $d > 0$?

(ii) Find the equation of the plane that is parallel to $\underline{r}.\underline{n} = d$ and contains the point $\underline{p}$.

(iii) Hence deduce the formula for the shortest distance from the point $\underline{p}$ to the plane $\underline{r}.\underline{n} = d$

Q16

Find the shortest distance between the point $(\mathbf{4}, \mathbf{-2}, \mathbf{3})$ and the line $\underline{\boldsymbol{r}} = \begin{pmatrix} \mathbf{7} \\ \mathbf{5} \\ \mathbf{-1} \end{pmatrix} + \boldsymbol{\lambda} \begin{pmatrix} \mathbf{3} \\ \mathbf{-6} \\ \mathbf{4} \end{pmatrix}$, leaving the answer in surd form.

Q18

Find the distance between the lines $\frac{\boldsymbol{x}+\mathbf{1}}{\mathbf{1}} = \frac{\boldsymbol{y}+\mathbf{2}}{\mathbf{2}}; \boldsymbol{z} = \mathbf{4}$ and $\frac{\boldsymbol{x}+\mathbf{3}}{\mathbf{1}} = \frac{\boldsymbol{y}-\mathbf{6}}{\mathbf{2}}; \boldsymbol{z} = \mathbf{7}$, leaving your answer in exact form.

Q19

(i) Show the lines $\frac{x-1}{2}=\frac{y+3}{5}=\frac{z-2}{3}$ and $\frac{x}{1}=\frac{y-4}{2}=\frac{z+1}{2}$ are skew.

(ii) Find the shortest distance between the lines and identify the points on the lines at which this shortest distance occurs.

Q22

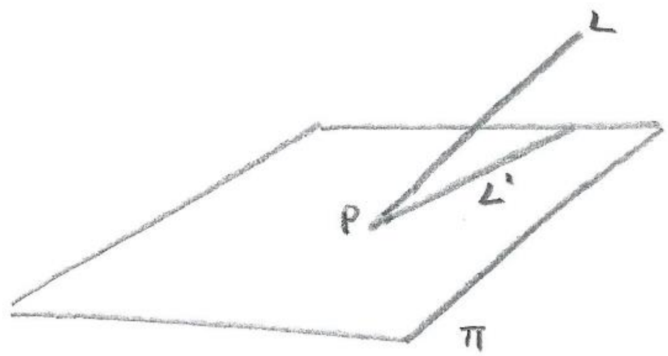
Show that the shortest distance from the point $\underline{p}$ to the plane

$\underline{r}.\underline{n}=d$ is $\frac{|d-\underline{p}.\underline{n}|}{|\underline{n}|}$

Q23

Given the plane Π: $3x+2y-z=6$ and the line

$L: \underline{r}=\begin{pmatrix}1\\0\\3\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\1\end{pmatrix}$, let L' be the projection of L onto Π



(i) Find the point of intersection (P) of Π & L

(ii) Find the angle between Π & L

(iii) Find a vector that is parallel to Π and perpendicular to L

(iv) Find a vector equation for L'

(v) Find the angle between L and L'

Vector Product

Q14

Find the volume of the tetrahedron with corners
$(2, 1, 3), (-1, 5, 0), (4, 4, 7), (8, 2, 2)$

Q21

Use the vector product to find the area of the triangle with corners A (1,2,3), B (4,5,6) & C (9,8,7)

Miscellaneous

Q4

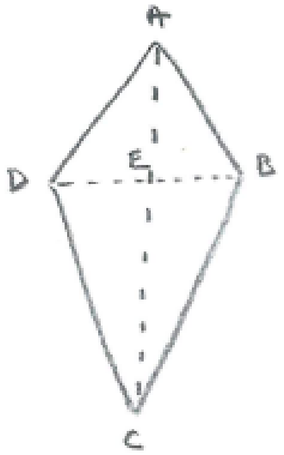
Find c, a & b such that $\begin{pmatrix} 2 \\ 3 \\ c \end{pmatrix} = a\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} + b\begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}$

[ie such that the 3 vectors are not linearly independent]

Q24

In the diagram below, ABCD is a kite. Find $\overrightarrow{OD}$ if $\overrightarrow{OA} = \begin{pmatrix} -1 \\ 4/3 \\ 7 \end{pmatrix}$,

$\overrightarrow{OB} = \begin{pmatrix} 4 \\ 4/3 \\ 2 \end{pmatrix}$ & $\overrightarrow{OC} = \begin{pmatrix} 6 \\ 16/3 \\ 2 \end{pmatrix}$



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